18.440 Practice Midterm Two: 50 minutes, 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (20 points) Let $X$ and $Y$ be independent Poisson random variables with parameter 1. Compute the following. (Give a correct formula involving sums — does not need to be in closed form.)

(a) The probability mass function for $X$ given that $X + Y = 5$.

(b) The conditional expectation of $Y^2$ given that $X = 2Y$.

(c) The probability mass function for $X - 2Y$ given that $X > 2Y$.

(d) The probability that $X = Y$.

2. (15 points) Solve the following:

(a) Let $X$ be a normal random variable with parameters $(\mu, \sigma^2)$ and $Y$ an exponential random variable with parameter $\lambda$. Write down the probability density function for $X + Y$.

(b) Compute the moment generating function and characteristic function for the uniform random variable on $[0, 5]$.

(c) Let $X_1, \ldots, X_n$ be independent exponential random variables of parameter $\lambda$. Let $Y$ be the second largest of the $X_i$. Compute the mean and variance of $Y$.

3. (10 points)

(a) Suppose that the pair $(X, Y)$ is uniformly distributed on the disc $x^2 + y^2 \leq 1$. Find $f_X$, $f_Y$.

(b) Find also $f_{X^2 + Y^2}$ and $f_{\max(x,y)}$.

(c) Find the conditional probability density for $X$ given $Y = y$ for $y \in [-1, 1]$.

(d) Compute $\mathbb{E}[X^2 + Y^2]$. 

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4. (10 points) Suppose that $X_i$ are independent random variables which take the values 2 and .5 each with probability 1/2. Let $X = \prod_{i=1}^{n} X_i$.

(a) Compute $EX$.

(b) Estimate the $P\{X > 100\}$ if $n = 100$.

5. (20 points) Suppose $X$ is an exponential random variable with parameter $\lambda_1 = 1$, $Y$ is an exponential random variable with $\lambda_2 = 2$, and $Z$ is an exponential random variable with parameter $\lambda_3 = 3$. Assume $X$ and $Y$ and $Z$ are independent and compute the following:

(a) The probability density function $f_{X+Y}$

(b) $\text{Cov}(XY, X + Y)$

(c) $\mathbb{E}\{\max\{X, Y, Z\}\}$

(d) $\text{Var}\{\min\{X, Y, Z\}\}$

(e) The correlation coefficient $\rho(\min\{X, Y, Z\}, \max\{X, Y, Z\})$.

6. (10 points) Suppose $X_1, \ldots, X_{10}$ be independent standard normal random variables. For each $i \in \{2, 3, \ldots, 9\}$ we say that $i$ is a local maximum if $X_i > X_{i+1}$ and $X_i > X_{i-1}$. Let $N$ be the number of local maxima. Compute

(a) The expectation of $N$.

(b) The variance of $N$.

(c) The correlation coefficient $\rho(N, X_1)$.

7. (15 points) Give the name and an explicit formula for the density or mass function of $\sum_{i=1}^{n} X_i$ when the $X_i$ are

(a) Independent normal with parameter $\mu, \sigma^2$.

(b) Independent exponential with parameter $\lambda$.

(c) Independent geometric with parameter $p$.

(d) Independent Poisson with parameter $\lambda$

(e) Independent Bernoulli with parameter $p$. 