18.440 Midterm 2 Solutions, Fall 2009

1.

(a) $P\{X < a\} = a^2$ for $a \in (0,1)$ and thus $F_X(a) = a^2$. Differentiating gives

$$f_X(a) = \begin{cases} 2a & a \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(b) $\mathbb{E}[X] = \int_0^1 f_X(x)x dx = \int_0^1 2x^2 dx = 2/3$.

(c) The pair (X,Y) is uniformly distributed on the triangle $\{(x,y) : x \in [0,1], y \in [0,1], x > y\}$. Thus, conditioned on X, Y is uniform on $[0,X]$, so $\mathbb{E}[Y|X] = X/2$.

(d) $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[A_1A_2] - \mathbb{E}[X](\mathbb{E}[1-X]) = 1/4 - 2/9 = 1/36$.

2.

(a) As derived in lecture and on problem sets, the collection of two Poisson processes (the earthquake and the flood processes) may be constructed equivalently by first taking a Poisson point process of rate $2+3=5$ and then independently declaring each point in the process to be an earthquake with probability $3/5$ (and a flood otherwise). Thus, the number N of earthquakes before the first flood satisfies $P\{N = k\} = (3/5)^k(2/5)$.

(b) One may recall that the sum of independent rate one exponentials is a gamma distribution with $\alpha = 2$ and $\lambda = 1$: the density is thus

$$f(t) = \begin{cases} e^{-\lambda t}t^{\alpha-1}\lambda^\alpha/\Gamma[\alpha] = e^{-t}t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

This can also be derived directly by letting X and Y denote the two exponential random variables and writing

$$f_{X+Y}(t) = \int_{x=0}^{t} f_X(x)f_Y(t-x)dx = \int_{x=0}^{t} e^{-x}e^{-(t-x)}dx = te^{-t}.$$

(c)

$$\mathrm{Cov}(\min\{E,F,M\},M) = \mathrm{Cov}(\min\{E,F,M\},\min\{E,F,M\})$$
$$+\mathrm{Cov}(\min\{E,F,M\},M-\min\{E,F,M\})$$

The memoryless property of exponentials implies that $\min\{E,F,M\}$ and $M-\min\{E,F,M\}$ are independent, so this becomes $\mathrm{Var}(\min\{E,F,M\})$. This is the variance of an exponential of rate 6, which is $1/36$.

3.

(a) $\mathbb{E}[X^2Y^2]-\mathbb{E}[XY]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2]-\mathbb{E}[X]^2\mathbb{E}[Y]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] = \frac{1}{3}$.

(b) $\mathbb{E}[e^{tX}e^{tY}] = \mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}] = e^{t^2/2}\frac{e^t-1}{t}$.

4.

(a) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(\sum_{i=1}^{60} X_i, \sum_{j=41}^{100} X_j) = \sum_{i=41}^{60}\mathrm{Cov}(X_i,X_i) = 20$. Since $\mathrm{Var}(X)=\mathrm{Var}(Y)=60$, we have $\rho(X,Y) = 20/\sqrt{60\cdot 60} = 1/3$.

(b) Let $Z=\sum_{i=61}^{100} X_i$. Then X and Z are independent with joint density $f(a,b) = \frac{1}{\sqrt{2\pi}\sqrt{60}}e^{-a^2/120}\frac{1}{\sqrt{2\pi}\sqrt{40}}e^{-b^2/80}$. Conditioning on $X+Z=x$ restricts us to the line $a+b=x$, so the conditional density for X will be (for some constant C)

$$f(a) = Cf(a,x-a) = Ce^{-a^2/120}e^{-(x-a)^2/80},$$

where C is chosen so that $\int_{-\infty}^{\infty} f(a)da = 1$.

5.

(a) If X_i is the multiplicative factor during the ith year, then $\mathbb{E}[X_i] = 2\cdot .4 + .5\cdot .6 = 1.1$ and $\mathbb{E}\prod X_i = 1.1^{100} \sim 13780.6$.

(b) We need to get at least 50 "up" steps. Mean number is 40, variance is $100(.6\cdot .4) = 24$. So, very roughly, the chance is $1-\Phi(10/\sqrt{24}) \sim .96$. Despite high expectation, investment usually loses money.