1. (20 points) Evaluate the following explicitly:

(a) \[\sum_{i=0}^{10} 9^i \frac{10!}{i!(10-i)!} = (1 + 9)^{10} = 10^{10}\] by the Binomial theorem.

(b) \[\sum_{i=5}^{9} 2^{-9} \frac{9!}{i!(9-i)!} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right)^9 = \frac{1}{2}\] by the Binomial theorem and \(\binom{9}{i} = \binom{9}{9-i}\).

2. (20 points) Six people, labeled \{1, 2, 3, 4, 5, 6\}, each own one hat. They throw their hats into a box, and each person removes and holds onto one of the six hats (with all of the 6! hat orderings being equally likely). Let \(M\) be the number of people who get their own hat. A pair of people is called a swapped pair if each one has the other person’s hat. For example, if person 1 has person 4’s hat and person 4 has person 1’s hat, then 1 and 4 constitute a single swapped pair. (There can be at most three swapped pairs.) Let \(S\) be the number of swapped pairs. Compute the following:

(a) \(\mathbb{E}[M]\) Let \(X_i\) be 1 if \(i\)th person gets own hat, 0 otherwise. Then

\[
\mathbb{E}[M] = \mathbb{E}\left[\sum_{i=1}^{6} X_i\right] = \sum_{i=1}^{6} \mathbb{E}X_i = \frac{6}{6} = 1
\]

(b) \(\mathbb{E}[S]\) Let \(X_{i,j}\) be 1 if \(i\) and \(j\) are a swapped pair, zero otherwise. Then

\[
\mathbb{E}[S] = \mathbb{E}\left[\sum_{1 \leq i < j \leq 6} X_{i,j}\right] = \binom{6}{2} \mathbb{E}X_{1,2} = \binom{6}{2} \frac{1}{5} \frac{1}{5} = \frac{1}{2}.
\]

(c) \(\text{Var}[M] = \sum_{i=1}^{6} \sum_{j=1}^{6} \text{Cov}[X_i, X_j] = 6\text{Cov}[X_1, X_1] + 30\text{Cov}[X_1, X_2] = 6 \frac{5}{36} + 30 \frac{1}{180} = 1\).

The first equality is a basic property of variance. The second equality comes from expanding the sum and collapsing symmetrically equivalent terms (e.g., \(\text{Cov}(X_1, X_1) = \text{Cov}(X_2, X_2)\)). We have \(\text{Cov}(X_1, X_1) = \text{Var}(X_1) = \frac{5}{36}\). Also,
\[
\text{Cov}(X_1, X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = \frac{1}{36} - \frac{1}{36} = \frac{1}{180}.
\]
3. (20 points) Let $D_1$ and $D_2$ be the outcomes (in $\{1, 2, 3, 4, 5, 6\}$) of two independent fair die rolls. Let $Y_i$ be the random variable which is equal to 1 if $D_1 = i$ and 0 otherwise. Compute the following:

(a) $E[D_2^2D_2^2] = E[D_2^2]E[D_2^2] = \frac{(91)^2}{36} = \frac{8281}{6}$ by independence and $E[D_2^2] = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$.

(b) $\text{Var}[D_1 - D_2] = \text{Var}[D_1] + \text{Var}[D_2] - 2\text{Cov}[D_1, D_2] = 2\text{Var}[D_1] = \frac{35}{6}$.

(c) $\text{Cov}(Y_1 + Y_2 + Y_3, Y_5 + Y_6) = 6\text{Cov}[Y_1, Y_5] = -\frac{1}{6}$ by bilinearity of covariance.

(d) $\text{Var}[\sum_{i=1}^{6} Y_i]$. The sum is constant, so the variance is zero.

4. (20 points) Let $X_1$, $X_2$, and $X_3$ be independent Poissonian random variables with parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, respectively. Compute the probabilities of the following events:

(a) The largest of $X_1$, $X_2$, and $X_3$ is at least 1. One minus the probability they are all zero is $1 - e^{-\lambda_1}e^{-\lambda_2}e^{-\lambda_3} = 1 - e^{-6}$.

(b) The largest of $X_1$, $X_2$, and $X_3$ is exactly 1. The probability that each $X_i$ is 1 or 0 is

$\prod_{i=1}^{3}(e^{-\lambda_i} + \lambda_i e^{-\lambda_i}) = e^{-\lambda_1\lambda_2\lambda_3}\prod_{i=1}^{3}(1 + \lambda_i) = 24e^{-6}$.

Subtracting the probability that the $X_i$ are all zero yields $23e^{-6}$.

5. (20 points) There are ten children: five attend school $A$, three attend school $B$, and two attend school $C$. Suppose that a pair of two children is chosen uniformly at random from the set of all possible pairs of children. Let $a$ be the number of students in the random pair that attend school $A$ and let $b$ be the number in the pair that attend school $B$. (So both $a$ and $b$ take values in the set $\{0, 1, 2\}$.)

(a) Compute $E[ab]$. Product will be non-zero (and equal to 1) only if $a = b = 1$. The expectation is the probability of this: $\frac{15}{\binom{10}{2}} = \frac{1}{3}$.

(b) Given that the two children in this pair attend the same school, what is the conditional probability that they both attend school $A$?

$\frac{\binom{5}{2}}{\binom{5}{2} + \binom{3}{2} + \binom{2}{2}} = \frac{10}{14} = \frac{5}{7}$. 

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