

18.600 Midterm 1, Fall 2026 Solutions

1. (10 points) Alice just entered an Uber car to begin a journey from Kendall Square to Logan Airport. At this time of day traffic is highly uncertain and she does not know how long the trip will take. Let X be the number of minutes her journey takes (rounded to the nearest 10). Alice believes that X will be a random element of $\{10, 20, 30, 40\}$ with all four numbers being equally likely. Compute the mean and variance of X . Give explicit numbers. **ANSWER:** Write $\mu = E[X] = \frac{1}{4}(10 + 20 + 30 + 40) = 25$. Then $(X - \mu)^2$ can be $25 = 5^2$ (prob $\frac{1}{2}$) or $225 = 15^2$ (prob $\frac{1}{2}$) so $\text{Var}(X) = E[(X - \mu)^2] = \frac{1}{2}(225 + 25) = 125$. Alternatively, $E[X^2] = \frac{1}{4}(10^2 + 20^2 + 30^2 + 40^2) = 750$ so $\text{Var}(X) = E[X^2] - E[X]^2 = 750 - 625 = 125$.

2. (20 points) Bob is fortunate enough to live in a New York City apartment directly over a shop that sells slices of cheese pizza for \$1. The shop is open from 9 am to midnight and Bob's diet consists entirely of cheese pizza. Bob works from home and his eating schedule is not altogether regular. During each of the 900 minutes that the shop is open, Bob decides (with probability $p = 1/150$, independently of all other minutes) to hop downstairs and buy a slice. Let X be the number of slices he will buy tomorrow.

- (a) Compute the probability mass function for X , along with $E[X]$ and $\text{Var}(X)$. Give exact expressions, not Poisson approximations. **ANSWER:** $P(X = k) = \binom{900}{k} \left(\frac{1}{150}\right)^k \left(\frac{149}{150}\right)^{900-k}$. Then $E[X] = np = 6$ and $\text{Var}(X) = npq = 6 \cdot \frac{149}{150} = \frac{149}{25}$.
- (b) Use a Poisson approximation to estimate the probability $P(X = 5)$. **ANSWER:** Take $\lambda = 6$ and write $e^{-\lambda}\lambda^k/k! = e^{-6}6^5/5!$.
- (c) Using the same Poisson approximation, estimate the conditional probability $P(X = 0|X < 2)$. Give an explicit rational number. **ANS:** $P(X = 0) \approx e^{-6}$ and $P(X = 1) \approx e^{-6}6^1/1! = 6e^{-6}$, so $P(X < 2) \approx e^{-6} + 6e^{-6} = 7e^{-6}$ and $P(X = 0|X < 2) \approx \frac{e^{-6}}{7e^{-6}} = \frac{1}{7}$.

3. (10 points) Later that day Bob decides to host his five best friends for a party. To prepare for the party, Bob purchases 32 slices of cheese pizza. How many ways are there to decide on a number of slices to give each of the six party participants, assuming that all 32 slices are used? In other words, how many non-negative integer sequences of the form $(a_1, a_2, \dots, a_6)$ satisfy $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 32$? **ANSWER:** Use stars and bars (5 bars, 32 stars) to get $\binom{37}{5} = \binom{37}{32}$.

4. (20 points) A certain job recruitment event contains 4 company booths and 4 candidates. There are three "random interview sessions" during which each candidate is randomly assigned to visit one of the 4 booths, with each booth getting exactly one candidate (and all 4! permutations being equally likely). These assignments are independent from one session to the next. Let E_i be the event that the i th company sees the *same candidate* during all three sessions. (When this happens, the company yields to "fate" and hires that candidate on the spot.)

- (a) Compute the probability $P(E_1)$. **ANSWER:** The 1st-session candidate has $\frac{1}{4}$ chance to come back 2nd session and (independently) $\frac{1}{4}$ to come back 3rd session, so answer is $\frac{1}{16}$.

- (b) Compute $P(E_1E_2)$ and $P(E_1E_2E_3)$ and $P(E_1E_2E_3E_4)$. **ANSWER:** Similar arguments give $P(E_1E_2) = \left(\frac{1}{4}\frac{1}{3}\right)^2 = \frac{1}{144}$ and $P(E_1E_2E_3) = P(E_1E_2E_3E_4) = \frac{1}{(4!)^2} = \frac{1}{576}$
- (c) Compute the probability at least one of these “fateful” events occurs. That is, compute $P(E_1 \cup E_2 \cup E_3 \cup E_4)$. (You can leave your answer as an alternating sum involving $\binom{n}{k}$ notation. No need to fully simplify.) **ANSWER:** Inclusion-exclusion and symmetry give $\binom{4}{1}P(E_1) - \binom{4}{2}P(E_1E_2) + \binom{4}{3}P(E_1E_2E_3) - \binom{4}{4}P(E_1E_2E_3E_4)$. FWIW this comes to $\frac{4}{16} - \frac{6}{144} + \frac{4}{576} - \frac{1}{576} = \frac{41}{192}$.

5. (20 points) Let’s think about the conditional probabilities associated to different possible carnival games.

- (a) A game involves tossing a fair coin independently 10 times. You win if all 10 are heads. What is the *conditional probability* that all 10 are heads *given* that the *first* 9 are heads? What is the *conditional probability* that all 10 are heads *given* that *at least* 9 (not necessarily consecutive) are heads? **ANSWER:** First answer is 1/2 by independence. For second answer, there are 10 ways to get one tails and 1 way to get 0 tails, so answer is 1/11.
- (b) At the next carnival booth over, there is an urn with 100 balls: 90 white and 10 black. You reach in and choose 10 at random, with all of the $\binom{100}{10}$ subsets being equally likely. Let E_k be the event that exactly k of your balls are black (and hence $10 - k$ are white). Compute $P(E_k)$ as a function of k for $k \in \{0, 1, \dots, 10\}$. **ANSWER:** $\frac{\binom{10}{k}\binom{90}{10-k}}{\binom{100}{10}}$.
- (c) Let A_k be the event that *at least* k balls are black. Compute the conditional probability $P(A_{10}|A_9)$. Give an exact rational number. **ANSWER:** There is one subset containing all 10 black balls and $\binom{10}{9}\binom{90}{1} = 900$ subsets containing exactly 9 black balls. Thus $P(A_{10}) = 1/\binom{100}{10}$ but $P(A_9) = 901/\binom{100}{10}$ and $P(A_{10}|A_9) = \frac{1}{901}$.

6. (10 points) Compute the following:

- (a) $\lim_{n \rightarrow \infty} \left(1 - \frac{8}{n}\right)^{3n}$ **ANSWER:** e^{-24}
- (b) $\sum_{k=0}^{\infty} \frac{7^k}{k!}$ **ANSWER:** e^7 .
- (c) $\sum_{k=0}^{10} \frac{10!}{(10-k)!} \cdot \frac{4^k}{k!}$ **ANSWER:** $(1+4)^{10} = 5^{10}$

7. (10 points) Jill is deciding randomly whether to study for her probability final. With probability $p \in (0, 1)$ she will study and with probability $(1 - p)$ she will play Minecraft all night instead. Let E be the event that Jill studies. Given that E occurs, the conditional probability that Jill gets an A in the class is .8. Given that E^c occurs, the conditional probability that Jill gets an A in the class is .1.

- (a) Compute the overall probability (as a function of p) that Jill gets an A in the class. **ANSWER:**
 $.8p + .1(1 - p) = .7p + .1$.
- (b) If we are given that Jill got an A in the class, what is the *conditional probability* (as a function of p) that she studied? **ANSWER:** $\frac{.8p}{.7p+.1} = \frac{8p}{7p+1}$.