Permutations, Patriots, and Powerball

18.600 Problem Set 1, due February 14

Welcome to your first 18.600 problem set! There will be ten problem sets this semester, each with a different theme, and each including a mix of problems of my own design and problems from the Sheldon Ross 8th edition textbook. Before we begin the problems, let me provide some basic information and experience-based advice to help you get more out of the course as a whole.

1. Recognize that, like courses in the 18.0x series, this is a fundamental course, meant to be accessible to students from every department at MIT. On the other hand (and here it is crucial to set expectations correctly) the course is *very challenging* for many students, and in some ways more advanced than the 18.0x courses. You should not be surprised or disappointed when you don't know how to do the problems right away, or when you need to consult a second or third source to understand a concept.

2. Attend lectures! I'll do my best to make lectures interactive and engaging, and I think we are somehow more connected if we can be synchronous when possible. People who stay engaged (ask questions, answer questions, etc.) learn more quickly, have more fun, and remember longer. If the lectures seem a little fast, try reading the textbook and/or slides in advance and come prepared to ask questions. If they seem a little slow (or if the topic is one you have seen before) try just showing up and letting the lecture be your first exposure. The fraction of students who have seen the lecture material before will be significant during the first few weeks, but will decline very quickly after that.

3. Use the textbook. Part of the reason for including problems from the textbook is to remind you that the textbook exists, and to encourage you to read it (any of the 6th through 10th editions will do). All of the course material (except for the parts about martingales and Black-Scholes, which will be covered in a separate handout) is covered in the textbook. There are many inexpensive ways to get electronic or hard copies of the textbook online (check out ebay, amazon, google, etc.)

4. Start the problem sets early and come to office hours if you have questions. The problem sets are more challenging than the exams and they serve a different purpose. They are meant to be educational in their own right. If you are one of the lucky few for whom the problems are easy, you will still learn a lot by thinking through the concepts and applications. You are free to collaborate with other students, look up material on the internet or in books, offer each other hints (though not full solutions or answers) on Piazza, and ask me and the TAs for ideas during office hours and recitations. Note however that some of the problems are reused from prior years, and you are definitely ***not* allowed to access or consult prior year problem sets or solutions, or to obtain answers from AI**. (Prior year

exam solutions, on the other hand, are posted on the public course webpage, and you are welcome to use those.)

5. If you are doing well in the course, try to help out by answering (as well as asking) questions on Piazza. This will help solidify your own understanding and will be appreciated by fellow students (as well as your TAs and me). I am going to try to restrain myself from answering most basic math questions on Piazza (so students have more of a chance to answer questions for each other) but if you post a question on Piazza and it remains unanswered after 48 hours, let me know by email and I'll look into it. If you have a personal problem or complaint or request (e.g., "Could you use a different chalk color?"), you should email the TAs or me directly, rather than posting something on Piazza, since things like this are awkward to address in a public forum.

6. Spare at least a *little* time for thinking and exploring that has nothing to do with your grade. Like looking up and reading a bit more about mathematical or practical issues raised in problem sets. Ponder some big questions about applications. What are we doing wrong in medicine? In traffic management? In college admissions? In teaching? In food preparation? Is there simple advice that, if followed, would make us all better off? Is there other commonly accepted advice that we should all stop following? How can we find out what these things are? How can probability help? As the course progresses, we will see many problems with applications; but each problem is the beginning of a conversation, not the end.

Plagiarism policy: We will abide by the MIT plagiarism policy (google *MIT plagiarism*) if a student consults AI or previous-year problem sets or previous-year official course solutions or solutions by previous-year students. There is a good chance we will be able to tell if this happens (e.g., if the AI—trained on earlier psets—uses numbers or formulas that appeared in those psets but not in the current year set) and it is a serious issue. But there are many legitimate (and enjoyable) ways to get pset help (collaboration with people you know, psetpartners.mit.edu, Piazza, office hours, recitations) so please use those instead! And you are welcome and encouraged to use the internet to supplement your textbook learning. (Wikipedia is often pretty good.)

Now the problems. The first problem set is about basic combinatorics. Cards, hats, permutations, balls, binomial and multinomial coefficients. It will help to keep these stories in mind as the course progresses. I have office hours after lecture on Fridays. Please stop by for discussion.

A. FROM ROSS 8th EDITION CHAPTER ONE:

1. **Theoretical Exercise 8:** Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \dots + \binom{n}{r}\binom{m}{0}.$$

   Hint: Consider a group of n men and m women. How many groups of size r are possible.

2. **Theoretical Exercise 13:** Show that, for $n > 0$, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

   In other words, show that the "sign-alternating sum" of the nth row of Pascal's triangle is equal to zero. (For example, when $n = 4$ we have $1 - 4 + 6 - 4 + 1 = 0$.)

B. Answer the following:

1. When we expand the product $(a + b + c + d)^8$, what is the coefficient in front of $a^3bc^2d^2$? Give the answer and a combinatorial explanation. After you are done, type `(a+b+c+d)^8` into wolframalpha.com or another computation package and check to see whether your answer was correct.

2. When we expand the product $(x^2 + x^3 + x^6 + x^7 + x^8)^{12}$, what is the coefficient in front of the term x^{53}? You can use wolframalpha.com or another computation package for this one.

3. Consider the following five ways to earn points in American football:

   Touchdown with no conversion: 6 points.

   Touchdown with one-point conversion: 7 points.

   Touchdown with two-point conversion: 8 points.

   Field goal: 3 points.

   Safety: 2 points.

   Imagine that Tom Brady, at age 53, decides to return to the New England Patriots to play in one final Super Bowl, and during the game his team scores exactly 53 points, through a sequence of 12 scoring events, each coming from the list above. How many ways are there for this to happen? In other words, how many sequences $a_1, a_2, \dots, a_{12}$ are there for which each a_i is an element of the set $\{6, 7, 8, 3, 2\}$ *and* $\sum_{i=1}^{12} a_i = 53$? Explain your answer. Is it related to a previous problem in this pset?

C. Imagine that the world has 24 time zones (evenly spaced around the globe) and that a class has exactly one student in each time zone.

1. How many ways to divide the students into 12 partnerships (2 students each) if you insist that each student's partner belong to a neighboring time zone, i.e., one of the zones where the time differs by ± 1 hour (ignoring date—so that the two timezones divided by the international date line are still considered neighbors).

2. How many ways are there to form 12 partnerships (2 students each) if you insist that each student's partner live in a timezone where the time differs by at most 2 hours (ignoring date)? **Hint:** This is a bit tricky and may involving summing over a few cases. Think about what "overlapping" partnerships might look like.

3. How many ways are there to divide the students into 12 groups of 2 if you don't care about timezones at all?

I first assigned this during the pandemic and my son recorded a "musical hint." See `https://math.mit.edu/~sheffield/2021600spring/pup.mp4`. I think this problem is bit trickier than the others, so definitely chat with people if you get stuck.

D. Consider permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. There are $n!$ such permutations altogether. Of these permulations...

1. How many fix all but three points? (In other words, how many have the property that $\sigma(j) = j$ for exactly $n - 3$ of the numbers $1, 2, \ldots, n$?)

2. How many have only one cycle, i.e., have the property that $\sigma(1), \sigma \circ \sigma(1), \sigma \circ \sigma \circ \sigma(1), \ldots$ cycles through all elements of $\{1, 2, \ldots, n\}$?

3. How many are involutions, i.e., have the property that for each j we have $\sigma \circ \sigma(j) = j$? (*Hint:* Argue that if σ is an involution then each j is either a fixed point — i.e., satisfies $\sigma(j) = j$ — or part of a cycle of length two. Compute the number of involutions with exactly k cycles of length 2, and then write your overall answer as a sum over k.)

4. In the special case that $n = 4$, how many permutations have no fixed points?

E. In the US lottery game of Powerball one is required to choose an (unordered) collection of five numbers from the set $\{1, 2, \ldots, 69\}$ (the white balls) along with another number from the set $\{1, 2, \ldots, 26\}$ (the red ball). So there are $\binom{69}{5} \cdot 26 = 292201338$ possible Powerball outcomes. You make your selection (five white, one red), the Powerball people choose theirs randomly, and you win if there is a match. Suppose that you have already chosen your numbers (the unordered set of five white, and the one red). How many possible Powerball outcomes match *exactly one* of your five white numbers (regardless of whether they match the red number)? How many match exactly two of your five white numbers? How many match exactly three? How many match one red ball *plus* exactly two white balls? Now, divide each of these numbers by 292201338 to produce a *probability* of seeing that outcome and use a calculator to give a numerical value. Write a sentence about what seems interesting or surprising about these values.

Remark: People in the US spend over \$100 billion per year on lottery tickets (about 32 percent of which is returned in big lottery payouts). That's about \$400 per adult, with many spending several thousand dollars a year. The psychological appeal may be hard for us to understand. But the fact that regular players get "close" now and then (matching two or three numbers) may be part of what keeps them coming back. If you are one of the *many* people who buys more than 1000 lottery tickets per year, you will probably match four of the six balls at some point during your life. If you have a dozen friends who do the same thing, one of them will probably match five of six balls at some point, which will *seem* very close. You can double check your computations by looking up the odds on the Powerball wikipedia page.

F. Derive the following formulas, which will be useful later in this course:

1. **Normal density formula:** $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = 1$. (Multiply both sides by $\sqrt{2\pi}$ and square both sides to get
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy = 2\pi.$$
Then derive this by computing the integral in polar coordinates. You can look up the derivation in the book if you get stuck.)

2. **Poisson mass formula:** $\sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k! = 1$ if $\lambda > 0$. Hint: recall (or look up) the Taylor expansion for the function $f(\lambda) = e^{\lambda}$.

3. **Binomial sum formula:** $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = 1$ where $p + q = 1$ and n is a positive integer. Hint: try expanding $(p + q)^n$.

4. **Factorial formula:** $\int_0^{\infty} x^n e^{-x} dx = n!$. (Assume $n \geq 0$ is an integer and use integration by parts and induction.)

Store these formulas in long term memory and write "Got it!"

Remark: You will at some point have to learn a few formulas for this class: in particular, those that appear in red on the so-called story sheet posted on the public course webpage. But it will turn out that a surprising number of them (perhaps a majority) are obtained in some way from the four formulas listed above. Internalizing these few facts now will help you a lot going forward. These formulas (or close variants) are among those appearing in a garageband clip `http://math.mit.edu/~sheffield/2018600/kindofthing.mp4` I posted a few years ago. The clip contains some nice identities and problems that you will see later in this course.