18.600 Midterm 2, Spring 2025: Solutions

1. (15 points) Aunt Nelly has four nieces. Starting from the current moment (time zero) each niece will phone Nelly on average once per week to ask for advice. For each niece, the times of those calls (measured in weeks) form a Poisson point process with parameter $\lambda = 1$. These Poisson point processes are independent from one niece to the next.

(a) Compute the probability that Nelly receives exactly 5 phone calls *total* from her nieces during the first week. **ANSWER:** Nelly expects 4 calls total; hence actual number is Poisson with parameter $\tilde{\lambda} = 4$. Probability that she gets $k = 5$ exactly is $e^{-\tilde{\lambda}}\tilde{\lambda}^k/k! = e^{-4}4^5/5!$.

(b) Let Y be the amount of time (in weeks) it takes until Nelly has received *at least one* phone call from *every* niece. In other words, if Y_i is the time until Nelly first receives a call from niece i then $Y = \max\{Y_i\}$. Compute the mean and variance of Y. **ANSWER:** The time until the first niece calls is exponential with parameter 4. Given that the subsequent time until first of the remaining three nieces calls is exponential with parameter 3. Similarly with parameters 2 and 1 for remaining two nieces. Using additivity of expectation (and of variance for independent random variables) and we find $E[Y] = 1/4 + 1/3 + 1/2 + 1$ and $\mathrm{Var}(Y) = 1/16 + 1/9 + 1/4 + 1$. This is the same as the "radioactive decay" problem, where one considers amount of time until 4 different particles have all decayed.

2. (15 points) Darlene is a competitive dart thrower. She aims to hit the bullseye, which is located at position $(0, 0)$. The place she actually hits the wall is the point (X, Y) which is X inches right and Y inches up from the bullseye. (Here X or Y can be negative if the dart is left of or below the bullseye.) The probability density function for the pair (X, Y) is given by $f(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$.

(a) Find the probability Darlene hits within an inch of the bullseye—i.e., find $P\big(X^2 + Y^2 \leq 1\big)$. **ANSWER:** We need to integrate $\frac{1}{2\pi}e^{-(x^2+y^2)/2}$ over the unit disc. This is easiest in polar coordinates where integral becomes

$$\int_0^\infty \int_0^{2\pi} \frac{1}{2\pi}e^{-r^2/2}rd\theta dr = \int_0^1 e^{-r^2/2}rdr = -e^{-r^2/2}|_0^1 = -1/\sqrt{e} - (-1) = 1 - 1/\sqrt{e} \approx .39.$$

(b) Suppose Darlene independently throws 16 darts. Let $(\overline{X}, \overline{Y})$ be the average of the 16 dart locations. Compute the probability that this average is less than a quarter inch from the bullseye. That is, compute $P(\overline{X}^2 + \overline{Y}^2 \leq 1/16)$. **ANSWER:** If $X_1, \ldots, X_{16}$ are i.i.d. instances of X then $\sum X_i$ is normal with mean 0, variance 16. Hence $\frac{1}{16}\sum X_i$ is normal with mean zero, variance 1/16, standard deviation 1/4. Similarly for Y. So $4(\overline{X}, \overline{Y})$ has the same law as (X, Y), so the answer is the same as the answer to (a).

3. (20 points) Veronica wants to be a veterinarian. She submits her application to ten *Schools of Veterinary Medicine*. Assume that there is uniform random variable $p \in [0, 1]$ that represents her application strength; given the value of p, each school accepts her (independently of all others) with probability p. To be clear: Veronica does not know p (it mostly depends on what her letter writers say) but her Bayesian prior on p is uniform. Each school has a different date on which it will tell all applicants whether they are accepted or rejected. **NOTE:** If it helps, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$.

(a) Suppose that Alice is rejected from the first two schools. Given this, what is her conditional probability density function for p? In other words, what is her Bayesian posterior for p after these observations. **ANSWER:** This beta with with $a - 1 = 0$ and $b - 1 = 2$, so $a = 1$ and $b = 3$. Hence density is $(1 - x)^2/B(a, b) = 3(1 - x)^2$.

(b) Given that Alice is rejected from the first two schools, what is the conditional probability that she is accepted to the third school? **ANSWER:** As shown on our problem set, this is expected value of p when p is chosen from the density computed in (a). Using the note, this comes to $a/(a + b) = 1/4$.

(c) Suppose that Alice receives more rejections, until as some point she has been rejected by five schools total and accepted to none. She begins to despair, but Bob tells her she still has a decent shot. *Given* Alice's five rejections, what is the *conditional probability* that she is accepted by *at least one* of the other five schools? **ANSWER:** By same argument as in (b), she conditionally has 6/7 chance to be rejected by 6th school; given that, she has 7/8 chance of being rejected from 7th school; given that 8/9 chance of being rejected by 8th school, etc. Conclude that (given first five rejections) she conditionally has

$$\frac{6}{7}\cdot\frac{7}{8}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\frac{10}{11}=\frac{6}{11}$$

chance to be rejected by all five remaining, hence 5/11 chance to be accepted by at least one.

4. (15 points) In Recycling City, there are two adjacent recycling bins: Bin A and Bin B. Each bin is meant for different type of material. However, the rules are sufficiently confusing that many people mixed them up. On the day after a marathon, 450 people come by with empty water bottles. Each person independently tosses a bottle in Bin A with probability 1/3 and Bin B with probability 2/3. Let X be the number of people who choose Bin A.

(a) Compute $E[X]$ and $\mathrm{Var}[X]$ and $\mathrm{SD}[X]$. **ANSWER:** X is binomial with $n = 450$ and $p = 1/3$. So $E[X] = np = 150$ and $\mathrm{Var}(X) = npq = 100$ and $\mathrm{SD}[X] = \sqrt{100} = 10$.

(b) Bin B will overflow if it is used by more than 320 people. Use the de Moivre-Laplace limit theorem to approximate the probability that more than 320 people choose Bin B. You may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$ in your answer. **ANSWER:** Number Y in Bin B has SD 10 and expectation 300. Chance to be at most 320 (2 SDs above mean) is approximately $\int_2^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1-\Phi(2) = \Phi(-2)$.

5. (15) Let $X_1, X_2, \ldots, X_5$ be i.i.d. random variables, each of which is uniform on $[0,1]$.

(a) Compute the probability $P(X_1 + X_2 < 1/2)$. **ANSWER:** The triangle of (X_1, X_2) values with $X_1 > 0$ and $X_2 > 0$ and $X_1 + X_2 < 1/2$ has area 1/8. Since density function is 1 on unit square, probability comes to 1/8.

(b) Compute the correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. **ANSWER:** If $\sigma^2 = \mathrm{Var}(X_i) = 1/12$ then $\mathrm{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)/\sqrt{\mathrm{Var}(X_1 + X_2 + X_3)\mathrm{Var}(X_2 + X_3 + X_4)} = 2\sigma^2/3\sigma^2 = 2/3$. The last step uses additivity of variance of independent random variables and the bilinearity of covariance.

(c) Compute the probability density function for $Z = \max\{X_1, X_2, X_3, X_4, X_5\}$. **ANSWER:** $F_Z(a) = P(Z \le a) = a^5$ for $a \in [0,1]$. Taking derivative gives $f_Z(x) = 5x^4$ for $x \in [0,1]$, and 0 for $x \notin [0,1]$.

6. (10 points) Harriet is cleaning out the dice cupboard at her gaming store when 100 six-sided dice fall onto the floor, each landing on an independently random value chosen uniform from $\{1,2,3,4,5,6\}$. Let Z be sum of the numbers on the 100 dice. Compute the moment generating function $M_Z(t)$. Give an exact formula. **ANSWER:** If X is roll on one die then $E[e^{tX}] = \frac{1}{6}\left(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right)$. Since we are summing n independent copies of X we raise this to the nth power: answer is

$$\left[\frac{1}{6}\left(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right)\right]^{100}.$$

7. (10 points) Suppose X has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$.

(a) Compute the expectation $E\left[(1+X^2)e^{-|X|}\right]$. **ANSWER:** If $g(X) = (1+X^2)e^{-|X|}$ then

$$E[g(X)] = \int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} e^{-|x|}/\pi = 2/\pi.$$

(b) Suppose X_1 through X_5 are i.i.d. random variables, *each* with probability density function $f(x) = \frac{1}{\pi(1+x^2)}$. Compute the probability $P(X_1 + X_2 > X_3 + X_4 + X_5 + 5)$. **ANSWER:** We can rewrite this as $P(\frac{X_1+X_2-X_3-X_4-X_5}{5} > 1)$. Note that $-X_i$ has same law as X_i (by symmetry of Cauchy) and using fact that average of independent Cauchy is Cauchy, this is same as $P(X > 1) = 1/4$.