18.600 Midterm 2, Spring 2025: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, notes or other resources may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).**

NAME: ____

1. (15 points) Aunt Nelly has four nieces. Starting from the current moment (time zero) each niece will phone Nelly on average once per week to ask for advice. For each niece, the times of those calls (measured in weeks) form a Poisson point process with parameter $\lambda = 1$. These Poisson point processes are independent from one niece to the next.

(a) Compute the probability that Nelly receives exactly 5 phone calls *total* from her nieces during the first week.

(b) Let Y be the amount of time (in weeks) it takes until Nelly has received *at least one* phone call from *every* niece. In other words, if Y_i is the time until Nelly first receives a call from niece i then $Y = \max\{Y_i\}$. Compute the mean and variance of Y.

2. (15 points) Darlene is a competitive dart thrower. She aims to hit the bullseye, which is located at position $(0,0)$. The place she actually hits the wall is the point (X,Y) which is X inches right and Y inches up from the bullseye. (Here X or Y can be negative if the dart is left of or below the bullseye.) The probability density function for the pair (X,Y) is given by $f(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$.

(a) Find the probability Darlene hits within an inch of the bullseye—i.e., find $P\big(X^2+Y^2 \leq 1\big)$.

(b) Suppose Darlene independently throws 16 darts. Let $(\overline{X},\overline{Y})$ be the average of the 16 dart locations. Compute the probability that this average is less than a quarter inch from the bullseye. That is, compute $P(\overline{X}^2+\overline{Y}^2 \leq 1/16)$.

3. (20 points) Veronica wants to be a veterinarian. She submits her application to ten *Schools of Veterinary Medicine*. Assume that there is uniform random variable $p \in [0, 1]$ that represents her application strength; given the value of p, each school accepts her (independently of all others) with probability p. To be clear: Veronica does not know p (it mostly depends on what her letter writers say) but her Bayesian prior on p is uniform. Each school has a different date on which it will tell all applicants whether they are accepted or rejected. **NOTE:** If it helps, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$.

(a) Suppose that Alice is rejected from the first two schools. Given this, what is her conditional probability density function for p? In other words, what is her Bayesian posterior for p after these observations.

(b) Given that Alice is rejected from the first two schools, what is the conditional probability that she is accepted to the third school?

(c) Suppose that Alice receives more rejections, until as some point she has been rejected by five schools total and accepted to none. She begins to despair, but Bob tells her she still has a decent shot. *Given* Alice's five rejections, what is the *conditional probability* that she is accepted by *at least one* of the other five schools?

4. (15 points) In Recycling City, there are two adjacent recycling bins: Bin A and Bin B. Each bin is meant for different type of material. However, the rules are sufficiently confusing that many people mixed them up. On the day after a marathon, 450 people come by with empty water bottles. Each person independently tosses a bottle in Bin A with probability $1/3$ and Bin B with probability $2/3$. Let X be the number of people who choose Bin A.

(a) Compute $E[X]$ and $\mathrm{Var}[X]$ and $\mathrm{SD}[X]$.

(b) Bin B will overflow if it is used by more than 320 people. Use the de Moivre-Laplace limit theorem to approximate the probability that more than 320 people choose Bin B. You may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.

5. (15) Let $X_1, X_2, \ldots, X_5$ be i.i.d. random variables, each of which is uniform on $[0,1]$.

(a) Compute the probability $P(X_1 + X_2 < 1/2)$.

(b) Compute the correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

(c) Compute the probability density function for $Z = \max\{X_1, X_2, X_3, X_4, X_5\}$.

6. (10 points) Harriet is cleaning out the dice cupboard at her gaming store when 100 six-sided dice fall onto the floor, each landing on an independently random value chosen uniform from $\{1, 2, 3, 4, 5, 6\}$. Let Z be sum of the numbers on the 100 dice. Compute the moment generating function $M_Z(t)$. Give an exact formula.

7. (10 points) Suppose X has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$.

(a) Compute the expectation $E\big[(1+X^2)e^{-|X|}\big]$.

(b) Suppose X_1 through X_5 are i.i.d. random variables, *each* with probability density function $f(x) = \frac{1}{\pi(1+x^2)}$. Compute the probability $P(X_1 + X_2 > X_3 + X_4 + X_5 + 5)$.