18.600 Midterm 1, Spring 2025: solutions

1. (20 points) The students at a rock-climbing class have to be divided into climbing partnerships containing two students each. If the class has 4 people (numbered 1,2,3,4) then there are three ways to do this: the partnerships can be $(1,2)$ and $(3,4)$. Or they can be $(1,3)$ and $(2,4)$. Or they can be $(1,4)$ and $(2,3)$. (To be clear, to specify a partnership division we only need to know who is paired with whom—we do not have to specify an ordering of the partnerships.)

(a) If a class has $2n$ students, let $K(n)$ be the number of ways to divide the students into n partnerships. Compute $K(n)$ for general $n \geq 1$. (Note that $K(2) = 3$ was given above.) **ANSWER:** There are $\binom{2n}{2,2,\ldots,2} = \frac{(2n)!}{(2!)^n} = (2n!)/2^n$ ways to group students into a first parternship, second partnership, etc. But since order doesn't matter, answer is $\frac{(2n)!}{2^n n!}$. Alternatively, imagine $2n$ students lined up left to right. Assign a partner for left student ($2n-1$ choices), then assign partner to leftmost unassigned student ($(2n-3)$ choices), and so forth. We find answer is $(2n-1)(2n-3)(2n-5)\cdot\ldots\cdot 3\cdot 1$. You can check that this is equivalent to $\frac{(2n)!}{2^n n!}$ using the fact that $2^n n! = 2n\cdot(2n-2)\cdot\ldots\cdot 2$.

(b) Suppose the class consists of n married couples, and one of the $K(n)$ ways of dividing the class into partnerships is chosen at random, with all such divisions being equally likely. Let X be the number of climbing partnerships whose members are married to each other. (So X is a random integer between 0 and n.) In case it helps, we let X_i be 1 if the members of the ith married couple are climbing partners and 0 otherwise. Compute (in terms of n) the expectation $E[X]$. **ANSWER:** A member of couple is assigned own spouse with probability $1/(2n-1)$. So $E[X_i] = 1/(2n-1)$ for each i. Hence $E[\sum X_i] = \sum E[X_i] = n/(2n-1)$.

(c) Assume $n \geq 2$ and compute the probability $P(X_1X_2 = 1)$. **ANSWER:** This is $P(E_1E_2)$ where E_i is event ith couple together. Answer is $P(E_1E_2) = P(E_1)P(E_2|E_1) = \frac{1}{2n-1}\cdot\frac{1}{2n-3} = \frac{1}{(2n-1)(2n-3)}$.

(d) Compute (in terms of n) the expectation $E[X^2]$. **ANSWER:** $E[X^2] = E[\sum_{i=1}^n X_i \sum_{j=1}^n X_j] = \sum_{i=1}^n\sum_{j=1}^n E[X_iX_j]$. Considering separately the n terms with $i=j$ and $n(n-1)$ terms with $i \neq j$, this becomes $\frac{n}{(2n-1)} + \frac{n(n-1)}{(2n-1)(2n-3)}$.

2. (10 points) A hungry group of 20 (distinguishable) students finds 400 (indistinguishable) bananas in the banana lounge and decides to consume them all. How many ways are there to divide the bananas among the students if we require that every student gets at least one banana? In other words, how many sequences $a_1, a_2, \ldots, a_{20}$ of *strictly positive* integers satisfy $\sum_{i=1}^{20} a_i = 400$. **ANSWER:** Every students gets one banana. Then apply stars and bars to remaining bananas: 380 stars and 19 bars. Answer is $\binom{399}{380} = \binom{399}{19}$.

3. (20 points) Harry is using an AI movie-maker to generate a video clip of himself water-ski-jumping over a shark. Each time the AI generates a clip, it is *awesome* with probability 1/500 (independently of what has happened before). Harry instructs the AI to generate 1000 clips in succession.

(a) Let A be the number of those clips that are awesome. Compute the probability $P(A = k)$ for $k \in \{0, 1, \ldots, 1000\}$. (Give an exact value, not a Poisson approximation.) **ANSWER:** A is binomial with $n = 1000$ and $p = 1/500$ and $q = 499/500$. Answer is $\binom{1000}{k}(1/500)^k(1/499)^{1000-k}$.

(b) Give exact formulas for the expectation $E[A]$ and the variance $\mathrm{Var}(A)$. **ANSWER:** $E[A] = np = 2$ and $\mathrm{Var}(A) = npq = 2\cdot(499/500) = 499/250$.

(c) Use Poisson approximations to estimate $P(A = 0)$ and $P(A = 1)$ and $P(A = 2)$. **ANSWER:** Taking $\lambda = 2$ have $P(A = k) \approx e^{-\lambda}\lambda^k/k! = \frac{2^k}{e^2 k!}$. Three cases are $\frac{1}{e^2}$ and $\frac{2}{e^2}$ and $\frac{2}{e^2}$.

(d) Use Poisson approximations to estimate the conditional probability $P(A = 2|A \leq 2)$. **ANSWER:** $P(A = 2, A \leq 2)/P(A \leq 2) = P(A = 2)/P(A \leq 2) = 2/5$.

4. (10 points) Compute the following:

(a) The coefficient in front of $x^2y^2z^2$ in the expansion of $(x+y+z)^6$. **ANSWER:** $\binom{6}{2,2,2} = 6!/(2!)^3$.

(b) $\sum_{k=0}^{\infty}\left(\frac{1}{2^k k!}\right)$ **ANSWER:** $e^{1/2}$, follows from Taylor expansion for e^x.

5. (10 points) When a customer walks into Darla's Diner, we denote by A the event the customer orders avocado toast, B the event the customer orders boba and C the event the customer orders a cheeseburger. Assume each item has a .5 chance of being ordered — i.e., $P(A) = P(B) = P(C) = 1/2$. Each pair of items has a 1/4 chance of being ordered together — i.e. $P(AB) = P(AC) = P(BC) = 1/4$. There is also a 1/4 probability that a customer orders all three, i.e., $P(ABC) = 1/4$. Given this information, compute the probability that the customer orders *at least one* of these three items. **ANSWER:** $P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) = 1$ by inclusion exclusion.

6. (15 points) Jill forms a lottery game in which there are 30 balls with labels $\{1, 2, \ldots, 30\}$, and somebody selects five of the balls at random, with all of the $\binom{30}{5}$ subsets being equally likely. Each lottery ticket has a printed list of 5 of the 30 numbers: for example, the numbers on a ticket could be $\{2, 5, 18, 26, 29\}$. The ticket wins the prize if its five numbers match those of the selected balls (the order in which balls are selected doesn't matter). Assume a lottery ticket is fixed before the ball selection begins.

(a) Compute the probability that *at least one* of the numbers on the lottery ticket is the number of one of the chosen balls. **ANSWER:** The probability of *no* matchings numbers is $\frac{\binom{25}{5}}{\binom{30}{5}} = \frac{25\cdot24\cdot23\cdot22\cdot21}{30\cdot29\cdot28\cdot27\cdot26}$ (which is about .37). Answer is 1 minus that (which is about .63).

(b) Compute the conditional probability that all five selected balls match numbers on the lottery ticket *given* that at least four of the selected balls match numbers on the lottery ticket. **ANSWER:** Let E_k be event exactly k match. Then $P(E_5) = 1/\binom{30}{5}$. For four balls to match, choose one of 5 on ticket to *not* match a ball, and one of remaining 25 balls for it to match to; find that $P(E_5) = 125/\binom{30}{5}$. Answer is $\frac{P(E_5)}{P(E_4)+P(E_5)} = \frac{1}{126}$. It is *much* harder to get 5 matches than 4 (even though getting *at least one* match is not so hard) so regular lottery players may *intuitively* feel like they are getting closer than they actually are.

7.(15 points) Let X be a the number obtained on a roll of a D4 die. In other words, X is a random element of $\{1, 2, 3, 4\}$ with all four numbers equally likely. Similarly, Y comes from an independent D20 roll, so Y is a random element of $\{1, 2, 3, \ldots, 20\}$ with all numbers being equally likely. Compute:

(a) Variance of X. **ANSWER:** $E[X^2] - E[X]^2 = (1 + 4 + 9 + 16)/4 - (5/2)^2 = (30 - 25)/4 = 5/4$.

(b) The expectation $E[(-1)^{XY}]$. **ANSWER:** The random variable $(-1)^{XY}$ takes value -1 with probability 1/4 (when X and Y are both odd, and hence XY is odd) and 1 with probability 3/4 (when XY is even). Hence answer is $\frac{1}{4}(-1) + \frac{3}{4} \cdot 1 = \frac{1}{2}$.

(c) The probability $P(X > Y)$. **ANSWER:** X can only be greater than Y is (X, Y) is $(2, 1)$ or $(3, 1)$ or $(3, 2)$ or $(4, 1)$ or $(4, 2)$ or $(4, 3)$. Hence answer is $6/80 = 3/40$.