18.600 Midterm 1, Spring 2025: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials and $\binom{n}{k}$ expressions — no need to multiply them out).**

NAME: ____

1. (20 points) The students at a rock-climbing class have to be divided into climbing partnerships containing two students each. If the class has 4 people (numbered 1,2,3,4) then there are three ways to do this: the partnerships can be $(1,2)$ and $(3,4)$. Or they can be $(1,3)$ and $(2,4)$. Or they can be $(1,4)$ and $(2,3)$. (To be clear, to specify a partnership division we only need to know who is paired with whom—we do not have to specify an ordering of the partnerships.)

(a) If a class has $2n$ students, let $K(n)$ be the number of ways to divide the students into n partnerships. Compute $K(n)$ for general $n \geq 1$. (Note that $K(2) = 3$ was given above.)

(b) Suppose the class consists of n married couples, and one of the $K(n)$ ways of dividing the class into partnerships is chosen at random, with all such divisions being equally likely. Let X be the number of climbing partnerships whose members are married to each other. (So X is a random integer between 0 and n.) In case it helps, we let X_i be 1 if the members of the ith married couple are climbing partners and 0 otherwise. Compute (in terms of n) the expectation $E[X]$.

(c) Assume $n \geq 2$ and compute the probability $P(X_1 X_2 = 1)$.

(d) Compute (in terms of n) the expectation $E[X^2]$.

2. (10 points) A hungry group of 20 (distinguishable) students finds 400 (indistinguishable) bananas in the banana lounge and decides to consume them all. How many ways are there to divide the bananas among the students if we require that every student gets at least one banana? In other words, how many sequences $a_1, a_2, \ldots, a_{20}$ of *strictly positive* integers satisfy $\sum_{i=1}^{20} a_i = 400$.

3. (20 points) Harry is using an AI movie-maker to generate a video clip of himself water-ski-jumping over a shark. Each time the AI generates a clip, it is *awesome* with probability $1/500$ (independently of what has happened before). Harry instructs the AI to generate 1000 clips in succession.

(a) Let A be the number of those clips that are awesome. Compute the probability $P(A = k)$ for $k \in \{0, 1, \ldots, 1000\}$. (Give an exact value, not a Poisson approximation.)

(b) Give exact formulas for the expectation $E[A]$ and the variance $\text{Var}(A)$.

(c) Use Poisson approximations to estimate $P(A = 0)$ and $P(A = 1)$ and $P(A = 2)$.

(d) Use Poisson approximations to estimate the conditional probability $P(A = 2 | A \leq 2)$.

4. (10 points) Compute the following:

(a) The coefficient in front of $x^2y^2z^2$ in the expansion of $(x + y + z)^6$.

(b) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{2^k k!}\right)$

5. (10 points) When a customer walks into Darla's Diner, we denote by A the event the customer orders avocado toast, B the event the customer orders boba and C the event the customer orders a cheeseburger. Assume each item has a .5 chance of being ordered — i.e., $P(A) = P(B) = P(C) = 1/2$. Each pair of items has a 1/4 chance of being ordered together — i.e. $P(AB) = P(AC) = P(BC) = 1/4$. There is also a 1/4 probability that a customer orders all three, i.e., $P(ABC) = 1/4$. Given this information, compute the probability that the customer orders *at least one* of these three items.

6. (15 points) Jill forms a lottery game in which there are 30 balls with labels $\{1, 2, \ldots, 30\}$, and somebody selects five of the balls at random, with all of the $\binom{30}{5}$ subsets being equally likely. Each lottery ticket has a printed list of 5 of the 30 numbers: for example, the numbers on a ticket could be $\{2, 5, 18, 26, 29\}$. The ticket wins the prize if its five numbers match those of the selected balls (the order in which balls are selected doesn't matter). Assume a lottery ticket is fixed before the ball selection begins.

(a) Compute the probability that *at least one* of the numbers on the lottery ticket is the number of one of the chosen balls.

(b) Compute the conditional probability that all five selected balls match numbers on the lottery ticket *given* that at least four of the selected balls match numbers on the lottery ticket.

7.(15 points) Let X be a the number obtained on a roll of a D4 die. In other words, X is a random element of $\{1, 2, 3, 4\}$ with all four numbers equally likely. Similarly, Y comes from an independent D20 roll, so Y is a random element of $\{1, 2, 3, \ldots, 20\}$ with all numbers being equally likely. Compute:

(a) The variance $\mathrm{Var}(X)$.

(b) The expectation $E[(-1)^{XY}]$.

(c) The probability $P(X > Y)$.