Fall 2025 18.600 Final Exam Solutions

1. (10 points) Alice is trapped in a malfunctioning robotaxi. The taxi was supposed to let her out at a Paris hotel, but instead it locked the doors and began making random visits to the city's most popular attractions, namely: **A:** Arc de Triomphe, **B:** Sacre-Coeur Basilica, **C:** Notre-Dame Cathedral, **D:** Louvre Pyramid and **E:** Eiffel Tower. Let $V_1, V_2, V_3, \dots$ be the locations of the taxi's successive visits. Assume V_1 is the Arc de Triomphe and subsequent V_i are chosen as follows:

A. After each visit to the Arc de Triomphe, the taxi always next visits the Sacre-Coeur Basilica.
B. After each visit to the Sacre-Coeur Basilica, the taxi always next visits Notre-Dame Cathedral.
C. After each visit to Notre-Dame Cathedral, the taxi next visits the Louvre Pyramid with probability 1/2 and the Eiffel Tower with probability 1/2.
D. After each visit to the Louvre Pyramid, the taxi always next visits the Eiffel Tower.
E. After each visit to the Eiffel Tower, the taxi always next visits the Arc de Triomphe.

Answer the following:

(a) Express the above transition rules as a Markov chain matrix M. **ANSWER:**

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Alice's phone battery runs out (as she attempts to reach customer support) and she realizes the taxi will make well over 20 visits before stopping. Find the probability that V_{21} is Arc de Triomphe. [**Hint:** What can happen between successive visits to the Arc de Triomphe?] **ANSWER:** The Markov chain is a "deterministic cycle" through the five sites, except that during each cycle there is a 1/2 chance of skipping the Louvre. Getting back to Arc de Triomphe takes 4 steps (if Louvre is skipped) or 5 steps (otherwise). The only way to write 20 as a sum of 4's and 5's is to take either five 4's or four 5's. So A_{21} is the Arc de Triomphe only if the Louvre is skipped none of the first four times (probability 1/16) or it is skipped all of the first five times (probability 1/32). Overall probability is $1/16 + 1/32 = 3/32$.

(c) What is the expected number of times that Alice visits Notre Dame before the first time she visits the Louvre Pyramid? **ANSWER:** Each visit to Notre Dame has a 1/2 chance of being followed by a Louvre visit (and the Louvre can only follow Notre Dame). Hence the number of visits to Notre Dame before first Louvre visit is a geometric random variable with parameter 1/2, and the expectation is 2.

(d) Over the long term, what fraction of Alice's visit are made to each of the five attractions? **ANSWER:** We want to solve

$$\begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D & \pi_E \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \pi_A & \pi_B & \pi_C & \pi_D & \pi_E \end{pmatrix}$$

These constraints imply $\pi_A = \pi_E = \pi_B = \pi_C$ and $\pi_D = \pi_C/2$. We also know that $\pi_A + \pi_B + \pi_C + \pi_D + \pi_E = 1$. Solving, we find $\pi_A = 2/9$, $\pi_B = 2/9$, $\pi_C = 2/9$, $\pi_D = 1/9$ and $\pi_E = 2/9$. This makes sense intuitively because the Markov chain essentially "cycles" through A, B, C, and E deterministically, while making a detour for D half the time.

2. (10 points) Ten lawyers apply to be *lead attorney* on a robotaxi lawsuit. Based on their written applications, the ith candidate is given a "score" A_i. Assume the A_i are i.i.d. uniform random variables on $[0,1]$. **NOTE:** If it helps, you may recall the fact that a Beta (a,b) random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a,b)$, where $B(a,b)=(a-1)!(b-1)!/(a+b-1)!$.

(a) Find the probability that exactly 6 of the 10 candidates have scores less than .6. **ANSWER:** Number is binomial with $p=.6$ and $n=10$. Probability it equals 6 is $\binom{10}{6}\cdot(.6)^6\cdot(.4)^4$.

(b) Give the probability density function for the third largest of the 10 scores. **ANSWER:** There are 7 scores lower than third largest, and 2 scores higher. So third largest is a beta random variable with $a-1=7$ and $b-1=2$ and has density $x^7(1-x)^2/B(8,3)=x^7(1-x)^2\frac{10!}{7!2!}$.

(c) Compute the correlation coefficient $\rho(A_1+A_2+A_3, A_2+A_3+A_4)$. **ANSWER:** By definition,

$$\rho(A_1+A_2+A_3, A_2+A_3+A_4)=\frac{\text{Cov}(A_1+A_2+A_3, A_2+A_3+A_4)}{\sqrt{\text{Var}(A_1+A_2+A_3)\text{Var}(A_2+A_3+A_4)}}.$$

Using additivity of covariance (and fact that covariance of independent random variables is 0) the top is $\text{Var}(A_1)+\text{Var}(A_2)=2\text{Var}(A_1)$. Recall (or quickly compute) that $\text{Var}(A_1)=1/12$ so the numerator is $2/12$. Similarly, the bottom is $\sqrt{(3/12)^2}=3/12$ and the hence ratio is $2/3$.

(d) The firm always puts the four highest scoring candidates on a "short list" of people to interview. Let A be the *average* of the scores of those top four candidates. Compute $E[A]$. **ANSWER:** The jth highest candidate corresponds to beta with $a=11-j$ and $b=j$. By given formula, this expectation is $a/(a+b)=(11-j)/11$. Averaging over $a\in\{1,2,3,4\}$ this is $(11-2.5)/11=17/22$.

3. (10 points) Let X be an exponential random variable with parameter $\lambda=1$. For each real number K write $C(K)=E[\max\{X-K,0\}]$.

(a) Compute $C(K)$ as a function of K for $K\geq 0$. **ANSWER:**

$$E[\max\{X-K,0\}]=\int_0^\infty e^{-x}\max\{x-K,0\}dx=\int_K^\infty e^{-x}(x-K)dx.$$

Translating the integral by K gives $\int_0^\infty e^{-(x+K)}(x+K-K)dx$. This simplifies to $=e^{-K}\int_0^\infty e^{-x}xdx=e^{-K}$.

(b) Compute the derivatives C' and C'' on $[0,\infty)$. **ANSWER:** Taking derivatives directly gives $C'(K)=-e^{-K}$ and $C''(K)=e^{-K}$. Alternatively, we can get the same answer by using the call function identities from lecture: $C'(K)=F_X(K)-1$ and $C''(K)=f_X(K)$.

(c) Compute the expectation $E[(X+1)^3]$. **ANSWER:** Recall that $E[X^k]=\int_0^\infty e^{-x}x^k dx=k!$. Thus $E[(X+1)^3]=E[X^3+3X^2+3X+1]=3!+3\cdot 2!+3\cdot 1!+1=6+6+3+1=16$.

(d) Compute the conditional probability $P(X>25|X>20)$. **ANSWER:** By memoryless property, this is same as $P(X>5)=e^{-5}$.

4. (10 points) Mikaela is a novice cross-country skier in difficult terrain. Every now and then she has either a *controlled fall* (safely toppling into soft snow) or a *massive wipe out* (losing control and tumbling several feet while sustaining bruises). The times $F_1,F_2,F_3,\ldots$ of the controlled falls form a Poisson process, with 3 such falls expected during any given hour. The times $W_1,W_2,\ldots$ of massive wipeouts form an independent Poisson process with 1 such wipeout expected during each hour.

(a) Compute the probability that there are exactly 5 massive wipeouts and 19 controlled falls during Mikaela's first 7 hours of skiing. **ANSWER:** This is $\left(e^{-\lambda_1}\lambda_1^{k_1}/k_1!\right)\left(e^{-\lambda_2}\lambda_2^{k_2}/k_2!\right)$ with $\lambda_1 = 7$ (7 expected massive wipeouts) and $k_1 = 5$ and $\lambda_2 = 21$ (21 expected controlled falls) and $k_2 = 19$. Plugging in numbers gives $\left(e^{-7}\cdot 7^5/5!\right)\left(e^{-21}\cdot 21^{19}/19!\right)$.

(b) Compute the density function for W_4 (the number of hours till the fourth wipeout). **ANSWER:** This is a gamma random variable with $\lambda = 1$ and $n = 4$. The density function is $e^{-x}x^3/3!$ on $[0,\infty)$.

(c) Compute the probability density function for the first fall of any kind, i.e. for $\min(F_1, W_1)$. **ANSWER:** The time until the first fall of any kind is exponential with parameter $3+1=4$, so the density is $4e^{-4x}$ on $[0,\infty)$.

(d) Find the probability that there are at least two wipeouts during the first hour. **ANSWER:** This is one minus the probability of zero or one wipeouts, which is $1 - e^{-1} - e^{-1} = 1 - 2/e$.

5. Ivanna decides to leave MIT and spend her life selling inexpensive ice cream product at the beach. Every customer at Ivanna's Ice Cream Stand rolls a fair four-sided die to decide how many dollars to spend. In other words, each customer spends \$1 with probability 1/4, and \$2 with probability 1/4, and \$3 with probability 1/4, and \$4 with probability 1/4. Let X_i be amount the ith customer spends, and assume the X_i are independent. Let $X = \sum_{i=1}^{80} X_i$ be the total amount spent by the first 80 customers. You may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$ in your answers to the following.

(a) Find the mean and variance of X_1. **ANSWER:** $E[X_1] = (1+2+3+4)/4 = 2.5$ and $E[X_1^2] = (1+4+9+16)/4 = 7.5$ so $\mathrm{Var}(X_1) = E[X_1^2] - E[X_1]^2 = 7.5 - (2.5)^2 = 1.25$.

(b) Find the mean, variance and standard deviation of X. **ANSWER:** $E[X] = 80E[X_1] = 200$ and $\mathrm{Var}(X) = 80\mathrm{Var}(X_1) = 100$ and standard deviation is 10.

(c) Ivanna needs at least \$180 to pay for her basic expenses. Use the central limit theorem to approximate the probability $P(X \geq 180)$. **ANSWER:** This is probability that X is larger than its "mean minus two standard deviations" and is approximately $1 - \Phi(-2) = \Phi(2)$.

(d) Compute the probability that exactly 20 customers spend each of the 4 possible dollar amounts (so 20 spend \$1 and 20 spend \$2 and 20 spend \$3 and 20 spend \$4). **ANSWER:** $\binom{80}{20,20,20,20}/4^{20} = \frac{80!}{(20!)^4 \cdot 4^{20}}$.

6. (10 points) Let the pair (X,Y) be uniformly distributed on the diamond-shaped region D of points (x,y) for which $|x| + |y| \leq 1$.

(a) Compute the probability density function $f(x,y)$ for the pair (X,Y). **ANSWER:** The region has area 2 so the density function is $f(x,y) = 1/2$ for $(x,y) \in D$ and 0 for $(x,y) \notin D$.

(b) Compute the expectation $E[X^2 + Y^2]$ **ANSWER:** By symmetry, we can compute the integral of X^2 over one quarter of D (and multiply by 8) so this is $8\int_0^1\int_0^{1-x}\frac{1}{2}x^2dydx = 4\int_0^1(1-x)x^2dx = 4\int_0^1(x^2 - x^3)dx = 4(1/3 - 1/4) = 1/3$.

(c) Compute $E[X^2|Y]$ as a function of Y. **ANSWER:** Given $Y = y$ (for $y \in [-1,1]$) the quantity X is conditionally uniform in $\left(-(1-|y|),(1-|y|\right)$ and hence has conditional variance (equal to conditional X^2 value since conditional mean is zero) given by $\left(2(1-|y|)\right)^2/12$. Hence answer is $(1-|Y|)^2/3$ for $Y \in [-1,1]$. (And undefined for $Y \notin [-1,1]$.)

(d) Compute the probability $P(X > 1/2)$. **ANSWER:** The area of the triangular subset of D for which $X > 1/2$ is $1/4$, so the probability is $(1/4)/2 = 1/8$.

7. (10 points) Carol invites four friends to her holiday complicated-board-game party. Let X_i be 1 if the ith friend attends and 0 otherwise, so that $X = X_1 + X_2 + X_3 + X_4$ is the total number of friends who show up. Assume that $X_1, X_2, \ldots, X_4$ are i.i.d. with each X_i equal to 1 with probability 1/2 and 0 with probability 1/2. Compute the following:

(a) The entropy $H[X]$. **ANSWER:** X is $0, 1, 2, 3, 4$ with probabilities $1/16, 4/16, 6/16, 4/16, 1/16$. Then $\sum p_i(-\log p_i) = 2 \cdot (1/16) \cdot 4 + 2 \cdot (1/4) \cdot 2 + (3/8) \cdot (3 - \log 3) = 21/8 - (3/8) \log 3$.

(b) The entropy $H[X_1 + 2X_2 + 4X_3 + 8X_4]$. **ANSWER:** The sum can take 16 possible values, all with equal probability, so answer is $-\log(1/16) = 4$.

(c) Assuming Carol invites friends in order, let Y be the number of friends she invites before the first friend accepts her invitation. In other words $Y = \min\{k : X_k = 1\}$ if at least one of the X_i is equal to 1. If $X_1 = X_2 = X_3 = X_4 = 0$ (so Carol has no friends accepting) then we formally write $Y = \infty$. Note that Y is a random variable taking values in $\{1, 2, 3, 4, \infty\}$. Find $H(Y)$. **ANSWER:** Y takes values $1, 2, 3, 4, \infty$ with probabilities $1/2, 1/4, 1/8, 1/16, 1/16$. So entropy is $(1/2) \cdot 1 + (1/4) \cdot 2 + (1/8) \cdot 3 + (1/16) \cdot 4 + (1/16) \cdot 4 = 15/8$.

(d) Describe an optimal strategy for guessing Y with the minimal expected number of yes-or-no questions. How many questions do you expect to need to ask with your strategy? **ANSWER:** Divide probability space in half each time. Ask "Did first accept?" If no, ask "Did second accept?" If no, ask "Did third accept?" If no, ask "Did fourth accept?" Expected number of questions is $H(Y) = 15/8$.

8. (10 points) Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, each equal to 0 with probability 1/8, 1 with probability 3/4 and 2 with probability 1/8. Write $S_n = \sum_{i=1}^n X_i$ and $A_n = S_n/n$.

(a) Compute the moment generating function $M_{S_{80}}(t)$ and moment generating function $M_{A_{80}}(t)$. **ANSWER:** $M_{X_1}(t) = E[e^{tX_1}] = \frac{1}{8} + \frac{3}{4}e^t + \frac{1}{8}e^{2t}$ so $M_{S_{80}}(t) = M_{X_1}(t)^{80} = \left(\frac{1}{8} + \frac{3}{4}e^t + \frac{1}{8}e^{2t}\right)^{80}$ and $M_{A_{80}}(t) = M_{S_{80}}(t/80) = \left(\frac{1}{8} + \frac{3}{4}e^{t/80} + \frac{1}{8}e^{2t/80}\right)^{80}$

(b) Compute $E[X_1]$ and $\text{Var}(X_1)$. **ANSWER:** $E[X_1] = 1$ and $\text{Var}(X_1) = 1/4$.

(c) Use the central limit theorem to approximate $P(S_{100} > 90)$. You may use the function $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** $E[S_{100}] = 100$ and $\text{Var}(S_{100}) = 100\text{Var}(X_1) = 25$ with standard deviation 5. Probability $S_{100} > 90$ is probability of being exceeding " mean minus two standard deviations" which is approximately $1 - \Phi(-2) = \Phi(2)$.

(d) Compute the correlation coefficient $\rho(S_{500}, S_{2000})$. **ANSWER:** $\frac{\text{Cov}((S_{500}, S_{2000})}{\sqrt{\text{Var}(S_{500})\text{Var}(S_{2000})}} = \frac{500(1/4)}{\sqrt{500(1/4) \cdot 2000(1/4)}} = 1/2$.

9. (10 points) Let $X_1, X_2, \ldots, X_{20}$ be i.i.d. random variables, each with density function $\frac{1}{\pi(1+x^2)}$.

(a) Find the probability that all the X_i lie in the interval $[-1, 1]$. Give an explicit number. **ANSWER:** Each X_i is Cauchy and lies in $[-1, 1]$ with probability 1/2 by the spinning flashlight argument. Using independence, answer is $(1/2)^{20}$.

(b) Compute the probability density function for the average $A = \frac{1}{20} \sum_{i=1}^{20} X_i$. **ANSWER:** Since these are Cauchy, the average has the same law as the original. Density is $\frac{1}{\pi(1+x^2)}$.

(c) Compute the probability density function for $Y = X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5$. **ANSWER:** Note that jX_j has the same law as the sum of j i.i.d. Cauchy random variables. So Y has the law of the sum of 15 i.i.d. Cauchy random variables, which in turn has the same law as $15X_1$. Hence the density is $\frac{1}{15\pi\left(1+(x/15)^2\right)}$

(d) Express $E[\cos(X_1 + X_2^2)]$ as a double integral. (You don't have to evaluate the integral.) **ANSWER:**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \cdot \frac{1}{\pi(1+y^2)} \cdot \cos(x+y^2)dxdy.$$

10. (10 points) Carl is thrilled to discover an opportunity to purchase standing room Bad Bunny tickets in Mexico City for only \$120. Unfortunately, Carl only has \$50. Carl decides to go to his local betting club, where he will make \$1 bets on fair coin tosses (so after each bet, his wealth goes up by \$1 with probability 1/2 and down by \$1 with probability 1/2). Let X_i be his wealth in dollars after i bets have taken place. (So in particular, $X_0 = 50$, and X_1 could be either 49 or 51, etc.) Let T be the *number* of bets Carl makes before his wealth reaches either 0 or 120.

(a) Which of the following random variable sequences (indexed by n) is a martingale? Just circle the appropriate letters:

(i) X_n **ANSWER:** Yes

(ii) X_n^2 **ANSWER:** No, can check e.g. that $E[X_1] \neq X_0$.

(iii) $X_n^2 - n$ **ANSWER:** Yes. If $M_n = X_n^2 - n$ then $E[M_{n+1}|\mathcal{F}_n] = E[X_{n+1}^2 - (n+1)|\mathcal{F}_n]$ which simplifies to $\frac{1}{2}\Big((X_n+1)^2 + (X_n-1)^2\Big) - n - 1 = X_n^2 - n = M_n$

(iv) $X_n^2 + n$ **ANSWER:** No, can check e.g. that $E[X_1] \neq X_0$.

(v) $(120 - X_n)X_n + n$ **YES:** This is the martingale 120 times (i) minus the martingale (iii).

(b) What is $P(X_T = 120)$? **ANSWER:** If p is the probability then $E[X_T] = 120p = X_0 = 50$ so $p = 5/12$.

(c) Compute the expectation $E[T]$. **ANSWER:** Use the martingale (v). At time T it is equal to X_T and at time zero it is equal to $70 \cdot 50 = 3500$. So $E[T] = 3500$.

(d) Compute the probability that Carl has an *epic comeback* — i.e., that his wealth gets all the way down to \$1 but then subsequently rises to \$120. **ANSWER:** The probability of reaching 1 before 120 is $70/119 = 10/17$. Given that he reaches 1, the probability he makes it back to \$120 before running out of money is 1/120. So answer is $\frac{10}{17} \cdot \frac{1}{120} = \frac{1}{12 \cdot 17} = \frac{1}{204}$.