Fall 2025 18.600 Final Exam: 100 points

Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

NAME: ______________________________

1. (10 points) Alice is trapped in a malfunctioning robotaxi. The taxi was supposed to let her out at a Paris hotel, but instead it locked the doors and began making random visits to the city's most popular attractions, namely: **A:** Arc de Triomphe, **B:** Sacre-Coeur Basilica, **C:** Notre-Dame Cathedral, **D:** Louvre Pyramid and **E:** Eiffel Tower. Let $V_1, V_2, V_3, \ldots$ be the locations of the taxi's successive visits. Assume V_1 is the Arc de Triomphe and subsequent V_i are chosen as follows:

A. After each visit to the Arc de Triomphe, the taxi always next visits the Sacre-Coeur Basilica.
B. After each visit to the Sacre-Coeur Basilica, the taxi always next visits Notre-Dame Cathedral.
C. After each visit to Notre-Dame Cathedral, the taxi next visits the Louvre Pyramid with probability 1/2 and the Eiffel Tower with probability 1/2.
D. After each visit to the Louvre Pyramid, the taxi always next visits the Eiffel Tower.
E. After each visit to the Eiffel Tower, the taxi always next visits the Arc de Triomphe.

Answer the following:

(a) Express the above transition rules as a Markov chain matrix M.

(b) Alice's phone battery runs out (as she attempts to reach customer support) and she realizes the taxi will make well over 20 visits before stopping. Find the probability that V_{21} is Arc de Triomphe. [**Hint:** What can happen between successive visits to the Arc de Triomphe?]

(c) What is the expected number of times that Alice visits Notre Dame before the first time she visits the Louvre Pyramid?

(d) Over the long term, what fraction of Alice's visit are made to each of the five attractions?

2. (10 points) Ten lawyers apply to be *lead attorney* on a robotaxi lawsuit. Based on their written applications, the ith candidate is given a "score" A_i. Assume the A_i are i.i.d. uniform random variables on $[0, 1]$. **NOTE:** If it helps, you may recall the fact that a Beta (a, b) random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a, b)$, where $B(a, b) = (a-1)!(b-1)!/(a+b-1)!$.

(a) Find the probability that exactly 6 of the 10 candidates have scores less than .6.

(b) Give the probability density function for the third largest of the 10 scores.

(c) Compute the correlation coefficient $\rho(A_1 + A_2 + A_3, A_2 + A_3 + A_4)$.

(d) The firm always puts the four highest scoring candidates on a "short list" of people to interview. Let A be the *average* of the scores of those top four candidates. Compute $E[A]$.

3. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$. For each real number K write $C(K) = E[\max\{X - K, 0\}]$.

(a) Compute $C(K)$ as a function of K for $K \geq 0$.

(b) Compute the derivatives C' and C'' on $[0, \infty)$.

(c) Compute the expectation $E[(X + 1)^3]$.

(d) Compute the conditional probability $P(X > 25 | X > 20)$.

4. (10 points) Mikaela is a novice cross-country skier in difficult terrain. Every now and then she has either a *controlled fall* (safely toppling into soft snow) or a *massive wipe out* (losing control and tumbling several feet while sustaining bruises). The times $F_1, F_2, F_3, \ldots$ of the controlled falls form a Poisson process, with 3 such falls expected during any given hour. The times $W_1, W_2, \ldots$ of massive wipeouts form an independent Poisson process with 1 such wipeout expected during each hour.

(a) Compute the probability that there are exactly 5 massive wipeouts and 19 controlled falls during Mikaela's first 7 hours of skiing.

(b) Compute the density function for W_4 (the number of hours till the fourth wipeout).

(c) Compute the probability density function for the first fall of any kind, i.e. for $\min(F_1, W_1)$.

(d) Find the probability that there are at least two wipeouts during the first hour.

5. Ivanna decides to leave MIT and spend her life selling inexpensive ice cream product at the beach. Every customer at Ivanna's Ice Cream Stand rolls a fair four-sided die to decide how many dollars to spend. In other words, each customer spends \$1 with probability 1/4, and \$2 with probability 1/4, and \$3 with probability 1/4, and \$4 with probability 1/4. Let X_i be amount the ith customer spends, and assume the X_i are independent. Let $X = \sum_{i=1}^{80} X_i$ be the total amount spent by the first 80 customers. You may use the function $\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answers to the following.

(a) Find the mean and variance of X_1.

(b) Find the mean, variance and standard deviation of X.

(c) Ivanna needs at least \$180 to pay for her basic expenses. Use the central limit theorem to approximate the probability $P(X \geq 180)$.

(d) Compute the probability that exactly 20 customers spend each of the 4 possible dollar amounts (so 20 spend \$1 and 20 spend \$2 and 20 spend \$3 and 20 spend \$4).

6. (10 points) Let the pair (X, Y) be uniformly distributed on the diamond-shaped region D of points (x, y) for which $|x| + |y| \leq 1$.

(a) Compute the probability density function $f(x, y)$ for the pair (X, Y).

(b) Compute the expectation $E[X^2 + Y^2]$

(c) Compute $E[X^2|Y]$ as a function of Y.

(d) Compute the probability $P(X > 1/2)$.

7. (10 points) Carol invites four friends to her holiday complicated-board-game party. Let X_i be 1 if the ith friend attends and 0 otherwise, so that $X = X_1 + X_2 + X_3 + X_4$ is the total number of friends who show up. Assume that $X_1, X_2, \ldots, X_4$ are i.i.d. with each X_i equal to 1 with probability 1/2 and 0 with probability 1/2. Compute the following:

(a) The entropy $H[X]$.

(b) The entropy $H[X_1 + 2X_2 + 4X_3 + 8X_4]$.

(c) Assuming Carol invites friends in order, let Y be the number of friends she invites before the first friend accepts her invitation. In other words $Y = \min\{k : X_k = 1\}$ if at least one of the X_i is equal to 1. If $X_1 = X_2 = X_3 = X_4 = 0$ (so Carol has no friends attending) then we formally write $Y = \infty$. Note that Y is a random variable taking values in $\{1, 2, 3, 4, \infty\}$. Find $H(Y)$.

(d) Describe an optimal strategy for guessing Y with the minimal expected number of yes-or-no questions. How many questions do you expect to need to ask with your strategy?

8. (10 points) Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, each equal to 0 with probability 1/8, 1 with probability 3/4 and 2 with probability 1/8. Write $S_n = \sum_{i=1}^n X_i$ and $A_n = S_n/n$.

(a) Compute the moment generating function $M_{S_{80}}(t)$ and moment generating function $M_{A_{80}}(t)$.

(b) Compute $E[X_1]$ and $\mathrm{Var}(X_1)$.

(c) Use the central limit theorem to approximate $P(S_{100} > 90)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer.

(d) Compute the correlation coefficient $\rho(S_{500}, S_{2000})$.

9. (10 points) Let $X_1, X_2, \ldots, X_{20}$ be i.i.d. random variables, each with density function $\frac{1}{\pi(1+x^2)}$.

(a) Find the probability that all the X_i lie in the interval $[-1, 1]$. Give an explicit number.

(b) Compute the probability density function for the average $A = \frac{1}{20}\sum_{i=1}^{20} X_i$.

(c) Compute the probability density function for $Y = X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5$.

(d) Express $E[\cos(X_1 + X_2^2)]$ as a double integral. (You don't have to evaluate the integral.)

10. (10 points) Carl is thrilled to discover an opportunity to purchase standing room Bad Bunny tickets in Mexico City for only \$120. Unfortunately, Carl only has \$50. Carl decides to go to his local betting club, where he will make \$1 bets on fair coin tosses (so after each bet, his wealth goes up by \$1 with probability 1/2 and down by \$1 with probability 1/2). Let X_i be his wealth in dollars after i bets have taken place. (So in particular, $X_0 = 50$, and X_1 could be either 49 or 51, etc.) Let T be the *number* of bets Carl makes before his wealth reaches either 0 or 120.

(a) Which of the following random variable sequences (indexed by n) is a martingale? Just circle the appropriate letters:

(i) X_n

(ii) X_n^2

(iii) $X_n^2 - n$

(iv) $X_n^2 + n$

(v) $(120 - X_n)X_n + n$

(b) What is $P(X_T = 120)$?

(c) Compute the expectation $E[T]$.

(d) Compute the probability that Carl has an *epic comeback* — i.e., that his wealth gets all the way down to \$1 but then subsequently rises to \$120.