Counting tricks and basic principles of probability

Discrete random variables
Outline

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Break “choosing one of the items to be counted” into a sequence of stages so that one always has the same number of choices to make at each stage. Then the total count becomes a product of number of choices available at each stage.
Selected counting tricks

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- Overcount by a fixed factor.
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- Answer: \( \binom{n}{n_1,n_2,\ldots,n_r} := \frac{n!}{n_1!n_2!\ldots n_r!} \).
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- Answer $\binom{n}{n_1,n_2,\ldots,n_r} := \frac{n!}{n_1!n_2!\ldots n_r!}$.
- How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?
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- How many sequences $a_1, \ldots, a_k$ of non-negative integers satisfy $a_1 + a_2 + \ldots + a_k = n$?
  - Answer: \( \binom{n+k-1}{n} \). Represent partition by $k-1$ bars and $n$ stars, e.g., as **|** || **|** **|** *.
Axioms of probability

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- Finite additivity: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$.
- Countable additivity: $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair $i$ and $j$. 
Consequences of axioms

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- \(P(AB) \leq P(A)\)
Observe \( P(A \cup B) = P(A) + P(B) - P(AB) \).

More generally, \( P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{1 < i_1 < i_2} P(E_{i_1} E_{i_2}) + \ldots + (-1)^{r+1} \sum_{1 < i_1 < \ldots < i_r} P(E_{i_1} \cap \ldots \cap E_{i_r}) = \ldots + (-1)^{n+1} P(E_1 \cap \ldots \cap E_n). \)

The notation \( P_{i_1 < i_2 < \ldots < i_r} \) means a sum over all of the \( \binom{n}{r} \) subsets of size \( r \) of the set \( \{1, 2, \ldots, n\} \).
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Also, \( P(E \cup F \cup G) = \)
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P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG).
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Famous hat problem

▶ \( n \) people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.

Inclusion-exclusion. Let \( E_i \) be the event that \( i \)th person gets own hat.

▶ What is \( P(\bigcup_{i=1}^{n} E_i) \)?

Answer:

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\frac{(n-r)!}{n!}
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There are \( \binom{n}{r} \) terms like that in the inclusion exclusion sum.

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1 - P(\bigcup_{i=1}^{n} E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \pm \frac{1}{n!} \approx 1/e \approx 0.36788
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Definition: $P(E|F) = \frac{P(EF)}{P(F)}$. 
Conditional probability

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- Call $P(E|F)$ the “conditional probability of $E$ given $F$” or “probability of $E$ conditioned on $F$”.

Nice fact:

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P(E_1E_2E_3\ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1E_2\ldots E_{n-1}).
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Useful when we think about multi-step experiments.

For example, let $E_i$ be event $i$th person gets own hat in the $n$-hat shuffle problem.
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Dividing probability into two cases

\[ P(E) = P(EF) + P(EF^c) \]
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In words: want to know the probability of \( E \). There are two scenarios \( F \) and \( F^c \). If I know the probabilities of the two scenarios and the probability of \( E \) conditioned on each scenario, I can work out the probability of \( E \).
Bayes’ theorem

Bayes’ theorem/law/rule states the following:

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- So \( P(A|B) \) is \( \frac{P(B|A)}{P(B)} \) times \( P(A) \).
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So \( P(A|B) \) is \( \frac{P(B|A)}{P(B)} \) times \( P(A) \).

Ratio \( \frac{P(B|A)}{P(B)} \) determines “how compelling new evidence is”.
$P(\cdot|F)$ is a probability measure

- We can check the probability axioms: $0 \leq P(E|F) \leq 1$, $P(S|F) = 1$, and $P(\cup E_i) = \sum P(E_i|F)$, if $i$ ranges over a countable set and the $E_i$ are disjoint.
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- The probability measure $P(\cdot|F)$ is related to $P(\cdot)$. 

To get former from latter, we set probabilities of elements outside of $F$ to zero and multiply probabilities of events inside of $F$ by $1/P(F)$. $P(\cdot)$ is the prior probability measure and $P(\cdot|F)$ is the posterior measure (revised after discovering that $F$ occurs).
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$P(\cdot)$ is the *prior* probability measure and $P(\cdot|F)$ is the *posterior* measure (revised after discovering that $F$ occurs).
Say $E$ and $F$ are independent if $P(EF) = P(E)P(F)$. 

▶ Equivalent statement: $P(E|F) = P(E)$.

▶ Also equivalent: $P(F|E) = P(F)$. 

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Say $E$ and $F$ are **independent** if $P(EF) = P(E)P(F)$.

Equivalent statement: $P(E|F) = P(E)$. Also equivalent: $P(F|E) = P(F)$. 
Independence of multiple events

Say $E_1 \ldots E_n$ are independent if for each
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\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots n\} \quad \text{we have}
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- In other words, the product rule works.

- Independence implies
  $P(E_1 E_2 E_3 | E_4 E_5 E_6) = P(E_1) P(E_2) P(E_3) P(E_4) P(E_5) P(E_6)$,
  and other similar statements.

- Does pairwise independence imply independence?
  No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.
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- Say $X$ is a **discrete** random variable if (with probability one) if it takes one of a countable set of values.
- For each $a$ in this countable set, write $p(a) := P\{X = a\}$. Call $p$ the **probability mass function**.
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Write $F(a) = P\{X \leq a\} = \sum_{x \leq a} p(x)$. Call $F$ the **cumulative distribution function**.
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Example: in $n$-hat shuffle problem, let $E_i$ be the event $i$th person gets own hat.
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Then $\sum_{i=1}^{n} 1_{E_i}$ is total number of people who get own hats.
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Expectation of a discrete random variable

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Represents weighted average of possible values $X$ can take, each value being weighted by its probability.
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Agrees with the \textbf{SUM OVER POSSIBLE X VALUES} definition:

$$E[X] = \sum_{x : p(x) > 0} xp(x).$$
If $X$ is a random variable and $g$ is a function from the real numbers to the real numbers then $g(X)$ is also a random variable.

$$E[g(X)] = \sum_{x} g(x) \cdot p(x).$$
Expectation of a function of a random variable

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Answer:

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If $X$ and $Y$ are distinct random variables, then

This is called the linearity of expectation.

Can extend to more variables
$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$. 

Additivity of expectation
If $X$ and $Y$ are distinct random variables, then
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Defining variance in discrete case

- Let $X$ be a random variable with mean $\mu$. 

Variance is one way to measure the amount a random variable "varies" from its mean over successive trials.

Very important alternate formula: 

$$\text{Var}[X] = E[X^2] - (E[X])^2.$$
Let $X$ be a random variable with mean $\mu$.

The variance of $X$, denoted $\text{Var}(X)$, is defined by $\text{Var}(X) = E[(X - \mu)^2]$. 

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Taking $g(x) = (x - \mu)^2$, and recalling that

$$E[g(X)] = \sum_{x: p(x) > 0} g(x)p(x),$$

we find that

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Uses the same units as $X$ itself.
If we switch from feet to inches in our "height of randomly chosen person" example, then $X$, $E[X]$, and $SD[X]$ each get multiplied by 12, but $Var[X]$ gets multiplied by 144.
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Bernoulli random variables

- Toss fair coin \( n \) times. (Tosses are independent.) What is the probability of \( k \) heads?

Answer:

\[
\frac{n^k}{2^n}
\]

What if coin has \( p \) probability to be heads?

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\[
\frac{n^k p^k (1 - p)^{n-k}}{n^k}
\]

Writing \( q = 1 - p \), we can write this as:

\[
\frac{n^k p^k q^{n-k}}{n^k}
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Can use binomial theorem to show probabilities sum to one:

\[
1 = \left( p + q \right)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}
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Number of heads is binomial random variable with parameters \((n, p)\).
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Let $X$ be a binomial random variable with parameters $(n, p)$. Here is one way to compute $E[X]$. 

Think of $X$ as representing number of heads in $n$ tosses of a coin that is heads with probability $p$. 

Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise. 

In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss. 

Note that $E[X_j] = p \cdot 1 + (1-p) \cdot 0 = p$ for each $j$. 

Conclude by additivity of expectation that $E[X] = \sum_{j=1}^{n} E[X_j] = np$. 

Decomposition approach to computing expectation
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Conclude by additivity of expectation that

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Compute variance with decomposition trick

\[ X = \sum_{j=1}^{n} X_j, \text{ so} \]
\[ E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_iX_j] \]
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- So \( E[X^2] = np + (n-1)np^2 = np + (np)^2 - np^2 \).
- Thus
  \[
  \]
- Can show generally that if \( X_1, \ldots, X_n \) independent then
  \[
  \text{Var}[\sum_{j=1}^{n} X_j] = \sum_{j=1}^{n} \text{Var}[X_j]
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Let $\lambda$ be some moderate-sized number. Say $\lambda = 2$ or $\lambda = 3$. Let $n$ be a huge number, say $n = 10^6$. 

A Poisson random variable $X$ with parameter $\lambda$ satisfies $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$. 

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Suppose I have a coin that comes on heads with probability $\lambda/n$ and I toss it $n$ times.
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Answer: \( np = \lambda \).
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Binomial formula:

$$\binom{n}{k} p^k (1 - p)^{n-k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} p^k (1 - p)^{n-k}.$$
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This is approximately $\frac{\lambda^k}{k!} (1 - p)^{n-k} \approx \frac{\lambda^k}{k!} e^{-\lambda}$. 

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$\Pr\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$. 
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Clever computation tricks yield $E[X] = \lambda$ and $\text{Var}[X] = \lambda$. 
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clever computation tricks yield $E[X] = \lambda$ and $\text{Var}[X] = \lambda$.

We think of a Poisson random variable as being (roughly) a Bernoulli $(n, p)$ random variable with $n$ very large and $p = \lambda/n$. 
A Poisson random variable $X$ with parameter $\lambda$ satisfies $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$ for integer $k \geq 0$.

Clever computation tricks yield $E[X] = \lambda$ and $\text{Var}[X] = \lambda$.

We think of a Poisson random variable as being (roughly) a Bernoulli $(n, p)$ random variable with $n$ very large and $p = \lambda/n$.

This also suggests $E[X] = np = \lambda$ and $\text{Var}[X] = npq \approx \lambda$. 
A Poisson point process is a random function $N(t)$ called a Poisson process of rate $\lambda$. 

- For each $t > s \geq 0$, the value $N(t) - N(s)$ describes the number of events occurring in the time interval $(s, t)$ and is Poisson with rate $(t - s)\lambda$.
- The numbers of events occurring in disjoint intervals are independent random variables.
- Probability to see zero events in first $t$ time units is $e^{-\lambda t}$.
- Let $T_k$ be time elapsed, since the previous event, until the $k$th event occurs. Then the $T_k$ are independent random variables, each of which is exponential with parameter $\lambda$. 
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More card problems

- What’s the probability of a full house in poker (i.e., in a five card hand, 2 have one value and three have another)?
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- What’s the probability of a full house in poker (i.e., in a five card hand, 2 have one value and three have another)?
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- What is the probability of a two-pair hand in poker?
- What is the probability of a bridge hand with 3 of one suit, 3 of one suit, 2 of one suit, 5 of another suit?
Probability have rare disease given positive result to test with 90 percent accuracy.
Disease problems

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- Say probability to have disease is $p$. 
Probability have rare disease given positive result to test with 90 percent accuracy.

Say probability to have disease is $p$.

$S = \{\text{disease, no disease}\} \times \{\text{positive, negative}\}$.
Disease problems

- Probability have rare disease given positive result to test with 90 percent accuracy.
- Say probability to have disease is $p$.
- $S = \{\text{disease, no disease}\} \times \{\text{positive, negative}\}$.
- $P(\text{positive}) = .9p + .1(1 - p)$ and $P(\text{disease, positive}) = .9p$. 
▶ Probability have rare disease given positive result to test with 90 percent accuracy.
▶ Say probability to have disease is $p$.
▶ $S = \{\text{disease, no disease}\} \times \{\text{positive, negative}\}$.
▶ $P(\text{positive}) = .9p + .1(1 - p)$ and $P(\text{disease, positive}) = .9p$.
▶ $P(\text{disease|positive}) = \frac{.9p}{.9p + .1(1 - p)}$. If $p$ is tiny, this is about $9p$. 