Entropy

- Suppose $X$ and $Y$ are independent random variables, each equal to 1 with probability 1/3 and equal to 2 with probability 2/3.
  - Compute the entropy $H(X)$.

- Compute $H(X+Y)$.
- Which is larger, $H(X+Y)$ or $H(X,Y)$? Would the answer to this question be the same for any discrete random variables $X$ and $Y$? Explain.
Suppose $X$ and $Y$ are independent random variables, each equal to 1 with probability $1/3$ and equal to 2 with probability $2/3$.

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Suppose \( X \) and \( Y \) are independent random variables, each equal to 1 with probability \( 1/3 \) and equal to 2 with probability \( 2/3 \).

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- Which is larger, \( H(X + Y) \) or \( H(X, Y) \)? Would the answer to this question be the same for any discrete random variables \( X \) and \( Y \)? Explain.
Entropy — answers

\[ H(X) = \frac{1}{3}(- \log \frac{1}{3}) + \frac{2}{3}(- \log \frac{2}{3}). \]
Entropy — answers

- $H(X) = \frac{1}{3}(-\log \frac{1}{3}) + \frac{2}{3}(-\log \frac{2}{3})$.
- $H(X + Y) = \frac{1}{9}(-\log \frac{1}{9}) + \frac{4}{9}(-\log \frac{4}{9}) + \frac{4}{9}(-\log \frac{4}{9})$. 
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- $H(X, Y)$ is larger, and we have $H(X, Y) \geq H(X + Y)$ for any $X$ and $Y$. To see why, write $a(x, y) = P\{X = x, Y = y\}$ and $b(x, y) = P\{X + Y = x + y\}$. Then $a(x, y) \leq b(x, y)$ for any $x$ and $y$, so
  
  $H(X, Y) = E[-\log a(x, y)] \geq E[-\log b(x, y)] = H(X + Y)$.
Markov chains

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Problem: describe towel-distribution evolution as a Markov chain and determine (over the long term) on what fraction of days Bob emerges from the shower to find no towel.
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After Bob showers, if there is at least one towel in the bathroom, Bob uses the towel and leaves it draped over a chair in the walk-in closet. If there is no towel in the bathroom, Bob grumpily goes to the walk-in closet, dries off there, and leaves the towel in the walk-in closet.

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When Alice showers, she first checks to see if at least one towel is present. If a towel is present, she dries off with that towel and returns it to the bathroom towel rack. Otherwise, she cheerfully retrieves both towels from the walk-in closet, then showers, dries off and leaves both towels on the rack.
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Let state 0, 1, 2 denote bathroom towel number.
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Shower state change Bob: 2 → 1, 1 → 0, 0 → 0.

Shower state change Alice: 2 → 2, 1 → 1, 0 → 2.

Morning state change AB: 2 → 1, 1 → 0, 0 → 1.

Morning state change BA: 2 → 1, 1 → 2, 0 → 2.

Markov chain matrix:

\[
M = \begin{bmatrix}
0 & 0.5 & 0.5 \\
0 & 1 & 0 \\
0 & 0.5 & 0.5
\end{bmatrix}
\]

Row vector \(\pi\) such that \(\pi M = \pi\) (with components of \(\pi\) summing to one) is:

\[
\begin{bmatrix}
\frac{2}{9} \\
\frac{4}{9} \\
\frac{1}{3}
\end{bmatrix}
\]

Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel \(\frac{2}{9} \times 1\frac{2}{9} = \frac{1}{3}\) of the time.
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2 \\
9 \\
4 \\
9 \\
1 \\
3
\end{pmatrix}
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- Bob finds no towel only if morning starts in state zero and Bob goes first. Over long term Bob finds no towel \( \frac{2}{9} \times \frac{1}{2} = \frac{1}{9} \) fraction of the time.
Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $1/2$ and $-1$ with probability $1/2$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the probability that $Y_n$ reaches $-25$ before the first time that it reaches $5$?
Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability $\frac{1}{2}$ and $-1$ with probability $\frac{1}{2}$. Let $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

- What is the probability that $Y_n$ reaches $-25$ before the first time that it reaches 5?
- Use the central limit theorem to approximate the probability that $Y_{9000000}$ is greater than 6000.
\[ p_{-25}25 + p_55 = 0 \text{ and } p_{-25} + p_5 = 1. \text{ Solving, we obtain } p_{-25} = 1/6 \text{ and } p_5 = 5/6. \]
\( p_{-25}25 + p_{5}5 = 0 \) and \( p_{-25} + p_{5} = 1 \). Solving, we obtain \( p_{-25} = \frac{1}{6} \) and \( p_{5} = \frac{5}{6} \).

One standard deviation is \( \sqrt{9000000} = 3000 \). We want probability to be 2 standard deviations above mean. Should be about \( \int_{2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \).
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Once you run out of properties you are out of the game. Winner is player who ends up with all the properties.
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Alice starts out thinking she has a 1/6 chance of winning. But then on her first turn she loses one of her properties (so she now has only three). Given this, what is her updated estimate of her probability of winning?
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What is the probability of an epic comeback—where the winner is a player who at some point had only 1 card?
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Find the probability at least one player has an epic collapse.
Number of cards Alice holds is a martingale. Her revised probability is $\frac{3}{24}=\frac{1}{8}$. 

The probability Alice has an epic comeback is $\left(\frac{20}{23}\right)\cdot \left(\frac{1}{24}\right)$. Summing over the six players gives $6\left(\frac{20}{23}\right)\cdot \left(\frac{1}{24}\right) = \frac{20}{92} = \frac{5}{23}$.

The probability for Alice is $\left(\frac{4}{23}\right)\cdot \left(\frac{1}{24}\right)$ and summing over six players gives $6\left(\frac{4}{23}\right)\cdot \left(\frac{1}{24}\right) = \frac{1}{23}$.

You might think to use inclusion-exclusion... but on inspection... it is impossible to two players to have an epic collapse. So the probability of an epic collapse is the same as the expected number of epic collapses.
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