**18.600 Midterm 2, Spring 2022: 50 minutes, 100 points**

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**
2. **No calculators, books, or notes may be used.**
3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

NAME: ___________________________________________________________________

1. (10 points) Alice is organizing a trivia night at her local pub, for which she needs 36 working pens. She looks in her various drawers and discovers that, over the years, she has accumulated 2500 pens. But many are old and dry so that each pen works with probability $\frac{1}{50}$ independently from one pen to the next. Let $X$ be the total number of working pens.

(a) Compute the mean and variance of $X$.

(b) Use a normal random variable to estimate the probability $P(X \geq 36)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer.

2. (15 points) Consider the "infinite corridor" $C = \{(x, y) : |y| \leq 1\}$. In other words, $C$ is the set of points in the plane that lie between the horizontal line $y = -1$ and the parallel line $y = 1$. Imagine that a student stands in the middle of the corridor at location $(0, 0)$. The student puts on a blindfold and spins around until the student is facing in a random direction (chosen uniformly from the full $2\pi$-radian range of possible angles). The student then walks in that direction, starting from $(0, 0)$, until reaching a point $(X, Y)$ on the boundary of $C$, where the student hangs a flier for an a capella concert. Note that $X$ is a random real number in the range $(-\infty, \infty)$ while $Y$ is a random integer with $P(Y = 1) = 1/2$ and $P(Y = -1) = 1/2$.

(a) Are $X$ and $Y$ independent? Explain in a sentence why or why not.

(b) Suppose 8 choir members independently follow the procedure above, and let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_8, Y_8)$ be the points on the boundary of $C$ that they reach. Let $(\overline{X}, \overline{Y}) := \frac{1}{8}\sum_{i=1}^{8}(X_i, Y_i)$ denote the average of this set of points. Compute $P(\overline{X} > 1)$.

(c) Compute $P(\overline{Y} = 0)$.

3. (20 points) Suppose that $X_1, X_2, \ldots$ are independent identically distributed random variables with each $X_i$ equal to 1 with probability 1/2 and 2 with probability 1/2. Write $A_n = \frac{1}{n}\sum_{i=1}^n X_i$. Compute each limit below, and alongside each answer write **SLLN** if the answer follows most directly from the strong law of large numbers, **WLLN** if it follows most directly from the weak law of large numbers, **CLT** if it follows most directly from the central limit theorem and **OTHER** if you are not using one of the above to derive the conclusion. If it helps you may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

(a) $\lim_{n\to\infty} P(A_n > 1.6)$

(b) $\lim_{n\to\infty} P(A_n = 1.5)$

(c) $P(\lim_{n\to\infty} A_n = 1.5)$

(d) $\lim_{n\to\infty} P(A_n > 1.5 + \frac{1}{\sqrt{n}})$

(e) $\lim_{n\to\infty} P(\sum_{j=1}^{n} X_j^2 > 2n)$

4. (15 points) Bob has Olivia Rodrigo's song "Good for you" stuck in his head. He is trying to focus on his Course 6 problem set, but every few minutes a line from that song comes to mind. The times $T_1, T_2, \ldots$ at which this occurs (measured in minutes, starting at time zero when Bob first begins to study) form a Poisson point process with parameter $\lambda = 1/3$. So the song pops into his head on average once every three minutes. Compute the following:

(a) The probability density function for $T_3$.

(b) The probability that the song pops into Bob's head *exactly* 20 times during the first hour of study.

(c) The expectation $\mathbb{E}[T_1^3]$.

5. (10 points) Alice, Bob, Carol, David and Eve are all taking a placement exam together. Denote their respective scores respectively by $A$, $B$, $C$, $D$ and $E$. They do not *a priori* know much about the exam or their preparation levels, and they view their scores as independent uniform random variables on $[0, 1]$. Alice then asks the test administrator what her class rank was (first, second, third, fourth or fifth) and is told that she was second highest—i.e., that $A$ is the second largest of the five random variables.

(a) Given this new information, give a revised probability density function $f_A$ for $A$ (i.e., a Bayesian posterior). **NOTE:** If you remember what this means, you may use the fact that a Beta $(a, b)$ random variable has expectation $a/(a+b)$ and density $x^{a-1}(1-x)^{b-1}/B(a, b)$, where $B(a, b) = (a-1)!(b-1)!/(a+b-1)!$.

(b) According to your Bayesian prior, the expected value of $A$ was $1/2$. *Given* that $A$ was the second largest of the random variables, what is your *revised expectation* of the value $A$?

6. (10 points) Suppose that the pair $(X, Y)$ is uniformly distributed on the triangle $T = \{(x, y) : 0 \le x, 0 \le y, 2x + y \le 2\}$. That is, the joint density function is given by

$$f_{X,Y}(x, y) = \begin{cases} 1 & (x, y) \in T \\ 0 & (x, y) \notin T \end{cases}.$$

(a) Compute the marginal density function $f_Y$.

(b) Compute the probability $P(X + Y < 1/2)$.

7. (20 points) Suppose that $X_1, X_2, X_3, X_4$ are independent exponential random variables, each with parameter $\lambda = 1$.

(a) Compute the the probability density function for $A = \min\{X_1, X_2, X_3, X_4\}$.

(b) Compute the variance of $B = \max\{X_1, X_2, X_3, X_4\}$.

(c) Write $C = X_1 + X_2 + X_3 + X_4$ and express the random variable $E[X_1 + X_2 + X_3|C]$ as a function of $C$.

(d) Compute the correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.