18.600 Midterm 1, Spring 2022 solutions

1. (20 points) 6 married couples—numbered 1 to 6—are preparing to board a small airplane. The plane has 6 rows of seats, with 2 seats per row. However, instead of seating one couple per row in the usual way, the airline is assigning seating randomly — with all 12! ways of assigning the 12 people to the 12 seats being equally likely. Let X be the number of married couples seated together (i.e. both spouses in the same row). Note that X is a random integer taking values between 0 and 6. Let E_i be the event that the ith couple is seated together.

(a) Compute the probabilities $P(E_1)$ and $P(E_2E_3)$. **ANSWER:** If Alice/Bob are Couple 1, then Alice is equally likely to sit by any of the 11 others, so $P(E_1) = 1/11$. If Alice/Bob are Couple 2 and Carol/Dave are Couple 3, then there is a $\frac{1}{11}$ chance Alice and Bob are together and *given* that, Carol is equally to sit by any of the other 9. Hence $P(E_2E_3) = \frac{1}{11} \cdot \frac{1}{9} = \frac{1}{99}$.

(b) Compute $P(E_4 \cup E_5)$. **ANSWER:** $P(E_4 \cup E_5) = P(E_4) + P(E_5) - P(E_4E_5) = \frac{2}{11} - \frac{1}{99} = \frac{17}{99}$ by inclusion-exlusion and (a).

(c) Compute $E[X]$. **ANSWER:** Set $X_i = 1_{E_i}$. By (a) $E[X_i] = 1/11$. $E[X] = \sum_{i=1}^{6} E[X_i] = 6/11$.

(d) Compute the expectation $E[X^2]$. **ANSWER:** $E[X_iX_j] = \begin{cases} 1/11 & i = j \\ 1/99 & i \neq j \end{cases}$ by (a). Then $E[X^2] = \sum_{i=1}^{6}\sum_{j=1}^{6} E[X_iX_j]$ has 6 terms with $i = j$ and 30 with $i \neq j$ so $E[X^2] = \frac{30}{99} + \frac{6}{11} = \frac{28}{33}$.

2.(20 points) Compute the following:

(a) $\sum_{k=0}^{\infty} \frac{1}{5^k \cdot k!}$ **ANSWER:** $e^{1/5}$ by Taylor expansion.

(b) $\lim_{n\to\infty} (1 - \frac{1}{3n})^{5n}$ **ANSWER:** $\lim_{n\to\infty} \left((1 - \frac{1}{3n})^{3n}\right)^{5/3} = e^{-5/3}$

(c) $\sum_{k=0}^{\infty} kq^{k-1}p$ where p and q are both positive with $p + q = 1$. **ANSWER:** This is the expectation of a geometric random variable with parameter p, which is $1/p$.

(d) $\sum_{k=0}^{20} \frac{20!}{k!(20-k)!}$ **ANSWER:** This is the binomial expansion of $(1 + 1)^{20} = 2^{20}$

3. (20 points) Alice loves basketball. She practices free throw shots at a court by the river. Each time she shoots, there is a 1/3 probability that she makes the shot and retrieves the ball, a 1/3 probability that she misses and retrieves the ball, and a 1/3 probability that she misses badly enough that the ball goes down the river and is lost forever. Alice starts the day with 10 basketballs and plans to keep shooting until all the balls are lost.

(a) What is the probability that Alice makes at least 10 shots before all 10 of her balls are lost? **ANSWER:** Imagine Alice keeps shooting until she *either* gets 10 river shots or 10 successful shots. By symmetry, her chance of getting the 10 successful shots first is 1/2.

(b) What is the probability that Alice's first 9 shots include 3 she makes, 3 she misses but retrieves, and 3 she loses to the river? **ANSWER:** $\binom{9}{3,3,3}/3^9 = \frac{9!}{(3!)^3 3^9}$

(c) Let X be the total number of shots Alice *takes* (whether she makes them or not) before all of her balls are lost forever. Compute $P(X=k)$ as a function of k. **ANSWER:** X is number of tosses of $p=1/3$ coin to get $r=10$ heads. Negative binomial: $P(X=k)=\binom{k-1}{r-1}p^r q^{k-r}$

(d) Compute $E[X]$. **ANSWER:** It takes $1/p=3$ shots on average to lose 1 ball. By additivity of expectation it takes 30 shots on average to lose 10 balls.

4. (15 points) Carol is taking a test to determine if she has a certain disease. She thinks *a priori* that she has a 1/2 chance of having the disease. If she has the disease, the test will return positive with probability 7/10, negative with probability 3/10. If she doesn't have the disease, the test will return positive with probability 1/10 and negative with probability 9/10.

(a) Compute the probability that the test result is positive. **ANSWER:** If $D=$ "has disease" and $T=$ "tests positive" then $P(T)=P(DT)+P(D^cT)=P(D)P(T|D)+P(D^c)P(T|D^c)=\frac{1}{2}\cdot\frac{7}{10}+\frac{1}{2}\cdot\frac{1}{10}=\frac{4}{10}$.

(b) Compute the conditional probability that Carol has the disease given that the test result is positive. **ANSWER:** $P(D|T)=P(DT)/P(T)=\frac{7/20}{4/10}=7/8$.

(c) Compute the conditional probability that Carol has the disease given that the test result is negative. **ANSWER:** $P(D|T^c)=P(DT^c)/P(T^c)=\frac{3/20}{6/10}=1/4$.

5. (10 points) Bob is an aspiring musician who uses a streaming service that gives him .5 cents every time somebody listens to his song. During each second of the day there is (independently of all other seconds) a probability of 1/3600 that somebody starts listening to his song that second. So on average one person listens to his song every hour.

(a) Bob has a lunch break from his day job in 3 hours, and he was hoping to buy a \$10 sandwich for lunch, but his cash account currently only has \$9.99. In order to buy the sandwich he needs to get at least 2 more listens during the next 3 hours. Use a Poisson approximation to estimate the probability that this happens. **ANSWER:** If X is Poisson with $\lambda=3$ and compute $P(X\geq 2)=1-P(X=0)-P(X=1)=1-e^{-3}\frac{3^0}{0!}-e^{-3}\frac{3^1}{1!}=1-4e^{-3}\approx .8$.

(b) Using the same Poisson approximation, estimate the mean and variance of the total number of listens during the next 10 hours. **ANSWER:** If X is Poisson with $\lambda=10$ then $E[X]=\mathrm{Var}(X)=10$.

6. (15 points) Answer the following:

(a) Let S_n be the number of heads in n independent tosses of a coin that comes up heads with probability p and tails with probability $q=1-p$. Compute the $n\to\infty$ limit of the probability that $\frac{S_n-np}{\sqrt{npq}}$ lies in the interval $(-1,1)$. You may use the function $\Phi(a)=\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ in your answer. **ANSWER:** This is $\Phi(1)-\Phi(-1)\approx .68$.

(b) Suppose that X is a random variable with probability density given by $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Compute the expectation $E[X^3+X+1]$. **ANSWER:** $E[X^3]=E[X]=0$ by symmetry, so this is $E[1]=1$.

(c) Suppose that X is a random variable with density

$$f_X(x)=\begin{cases}x/2 & x\in[0,2]\\ 0 & x\notin[0,2]\end{cases}.$$

Compute the probability $P(X>1)$. **ANSWER:** $\int_1^2(x/2)dx=3/4$.