Outline

- Moment generating functions
- Characteristic functions
- Continuity theorems and perspective
Moment generating functions

Characteristic functions

Continuity theorems and perspective
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When $X$ is discrete, can write $M(t) = \sum_x e^{tx} p_X(x)$. So $M(t)$ is a weighted average of countably many exponential functions.

We always have $M(0) = 1$.

If $b > 0$ and $t > 0$ then $E[e^{tX}] \geq E[e^{t\min\{X, b\}}] \geq P\{X \geq b\} e^{tb}$.

If $X$ takes both positive and negative values with positive probability then $M(t)$ grows at least exponentially fast in $|t|$ as $|t| \to \infty$. 


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Also $M''(t) = d/dt M'(t) = d/dt E[Xe^{tX}] = E[X^2e^{tX}]$. So $M''(0) = E[X^2]$. Same argument gives that the $n$th derivative of $M$ at zero is $E[X^n]$.

Interesting: knowing all of the derivatives of $M$ at a single point tells you the moments $E[X^k]$ for all integer $k \geq 0$. Another way to think of this: write $e^{tX} = 1 + tX + t^2X^2/2! + t^3X^3/3! + \ldots$. Taking expectations gives $E[e^{tX}] = 1 + tm_1 + t^2m_2/2! + t^3m_3/3! + \ldots$, where $m_k$ is the $k$th moment. The $k$th derivative at zero is $m_k$. 

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Let $X$ and $Y$ be independent random variables and $Z = X + Y$. If you knew $M_X(t)$ and $M_Y(t)$, could you compute $M_Z(t)$? By independence, $M_Z(t) = E[e^{t(X+Y)}] = E[e^{tX}] E[e^{tY}] = M_X(t) M_Y(t)$ for all $t$. In other words, adding independent random variables corresponds to multiplying moment generating functions.
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Write the moment generating functions as $M_X(t) = E[e^{tX}]$ and $M_Y(t) = E[e^{tY}]$ and $M_Z(t) = E[e^{tZ}]$. If you knew $M_X$ and $M_Y$, could you compute $M_Z$?

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Moment generating functions for sums of i.i.d. random variables

We showed that if $Z = X + Y$ and $X$ and $Y$ are independent, then $M_Z(t) = M_X(t)M_Y(t)$.
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If $X_1 \ldots X_n$ are i.i.d. copies of $X$ and $Z = X_1 + \ldots + X_n$ then what is $M_Z$?
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Answer: $M^n_X$. Follows by repeatedly applying formula above.

This a big reason for studying moment generating functions. It helps us understand what happens when we sum up a lot of independent copies of the same random variable.
Other observations

- If $Z = aX$ then can I use $M_X$ to determine $M_Z$?

  - **Answer:** Yes. $M_Z(t) = E[e^{tZ}] = E[e^{tX}] = M_X(at)$.

- If $Z = X + b$ then can I use $M_X$ to determine $M_Z$?

  - **Answer:** Yes. $M_Z(t) = E[e^{tZ}] = E[e^{tX} + bt] = e^{bt}M_X(t)$.

  - The latter answer is the special case of $M_Z(t) = M_X(t)M_Y(t)$ where $Y$ is the constant random variable $b$. 
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Let’s try some examples. What is $M_X(t) = E[e^{tX}]$ when $X$ is binomial with parameters $(p, n)$? Hint: try the $n = 1$ case first.
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What if $X$ is Poisson with parameter $\lambda > 0$?

Answer: $M_X(t) = E[e^{tx}] = \sum_{n=0}^{\infty} \frac{e^{tn}e^{-\lambda}\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda} e^{\lambda e^t} = \exp[\lambda(e^t - 1)]$. 
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We know that if you add independent Poisson random variables with parameters \( \lambda_1 \) and \( \lambda_2 \) you get a Poisson random variable of parameter \( \lambda_1 + \lambda_2 \). How is this fact manifested in the moment generating function?
More examples: normal random variables

- What if $X$ is normal with mean zero, variance one?

  \[ M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} \, dx = e^{t^2/2}. \]

- What does that tell us about sums of i.i.d. copies of $X$?

  If $Z$ is sum of $n$ i.i.d. copies of $X$, then $M_Z(t) = e^{nt^2/2}$.

- What is $M_Z$ if $Z$ is normal with mean $\mu$ and variance $\sigma^2$?

  $Z$ has the same law as $\sigma X + \mu$, so

  \[ M_Z(t) = M(\sigma t) e^{\mu t} = e^{\sigma^2 t^2/2 + \mu t}. \]
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- What if $Z$ is a Γ distribution with parameters $\lambda > 0$ and $n > 0$?
- Then $Z$ has the law of a sum of $n$ independent copies of $X$.
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Seems that unless $f_X(x)$ decays superexponentially as $x$ tends to infinity, we won’t have $M_X(t)$ defined for all $t$. 

Informal statement: moment generating functions are not defined for distributions with fat tails.
More examples: existence issues

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- What is $M_X$ if $X$ is standard Cauchy, so that $f_X(x) = \frac{1}{\pi(1+x^2)}$. 

Answer: $M_X(0) = 1$ (as is true for any $X$) but otherwise $M_X(t)$ is infinite for all $t \neq 0$. 

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Outline

Moment generating functions

Characteristic functions

Continuity theorems and perspective
Outline

Moment generating functions

Characteristic functions

Continuity theorems and perspective
Let $X$ be a random variable.

The characteristic function of $X$ is defined by

$$\phi(t) = \phi_X(t) := E[e^{itX}].$$

Like $M(t)$ except with $i$ thrown in.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic functions are similar to moment generating functions in some ways.

For example, $\phi_X + \phi_Y = \phi_X \cdot \phi_Y$, just as $M_X + M_Y = M_X \cdot M_Y$.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_{X}(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

But characteristic functions have a distinct advantage: they are always well defined for all $t$ even if $f_X$ decays slowly.
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- Continuity theorems and perspective
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Characteristic functions are Fourier transforms of the corresponding distribution density functions and encode “periodicity” patterns. For example, if \( X \) is integer valued, \( \phi_X(t) = E[e^{itX}] \) will be 1 whenever \( t \) is a multiple of \( 2\pi \).
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We say that $X_n$ converge in distribution or converge in law to $X$ if $\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$ at all $x \in \mathbb{R}$ at which $F_X$ is continuous.
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Moment generating analog: if moment generating functions $M_{X_n}(t)$ are defined for all $t$ and $n$ and $\lim_{n \to \infty} M_{X_n}(t) = M_X(t)$ for all $t$, then $X_n$ converge in law to $X$. 