18.600: Lecture 24

Covariance and some conditional expectation exercises

Scott Sheffield

MIT
Outline

Covariance and correlation

Paradoxes: getting ready to think about conditional expectation
Covariance and correlation

Paradoxes: getting ready to think about conditional expectation
A property of independence

If $X$ and $Y$ are independent then

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$. 

Since $f(x, y) = f_X(x)f_Y(y)$ this factors as

$\int_{-\infty}^{\infty} h(y)f_Y(y)dy \int_{-\infty}^{\infty} g(x)f_X(x)dx = E[h(Y)]E[g(X)]$. 

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- Just write 
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Covariance (like variance) can also be written a different way. Write $\mu_X = E[X]$ and $\mu_Y = E[Y]$. If laws of $X$ and $Y$ are known, then $\mu_X$ and $\mu_Y$ are just constants.
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- Note: if $X$ and $Y$ are independent then $\text{Cov}(X, Y) = 0$. 
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- **General statement of bilinearity of covariance:**

$$\text{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \text{Cov}(X_i, Y_j).$$
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- **Special case:**
  \[
  \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{(i,j):i<j} \text{Cov}(X_i, X_j).
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Correlation doesn't care what units you use for $X$ and $Y$. If $a > 0$ and $c > 0$ then $\rho(aX + b, cY + d) = \rho(X, Y)$. 

Satisfies $-1 \leq \rho(X, Y) \leq 1$. Why is that? Something to do with $E[(X + Y)^2] \geq 0$ and $E[(X - Y)^2] \geq 0$?

If $a$ and $b$ are constants and $a > 0$ then $\rho(aX + b, X) = 1$. 

If $a$ and $b$ are constants and $a < 0$ then $\rho(aX + b, X) = -1$. 

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Say $X$ and $Y$ are uncorrelated when $\rho(X, Y) = 0$. 

Are independent random variables $X$ and $Y$ always uncorrelated?

Yes, assuming variances are finite (so that correlation is defined).

Are uncorrelated random variables always independent?

No. Uncorrelated just means $E[(X - E[X])(Y - E[Y])] = 0$, i.e., the outcomes where $(X - E[X])(Y - E[Y])$ is positive (the upper right and lower left quadrants, if axes are drawn centered at $(E[X], E[Y])$) balance out the outcomes where this quantity is negative (upper left and lower right quadrants). This is a much weaker statement than independence.
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Important point

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  - Yes, assuming variances are finite (so that correlation is defined).
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- Compute the correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. 

- Can we generalize this example?

- What is variance of number of people who get their own hat in the hat problem?

- Define $X_i$ to be 1 if $i$th person gets own hat, zero otherwise.

- Recall formula $\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)$.

- Reduces problem to computing $\text{Cov}(X_i, X_j)$ (for $i \neq j$) and $\text{Var}(X_i)$. 

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- Fairly dark as math humor goes (and no offense intended to anyone...) but dilemma is interesting.
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Variant without probability: Stay in hell for $n$ (of your choice) days, and thereafter on days that are multiples of $2^n$.

Another variant: infinitely many identical money sacks with labels 1, 2, 3, ... I have sack 1. You have all others.

You offer me a deal. I give you sack 1, you give me sacks 2 and 3. I give you sack 2 and you give me sacks 4 and 5. On the $n$th stage, I give you sack $n$ and you give me sacks $2n$ and $2n+1$. Continue until I say stop.

Let me get arbitrarily rich. But if I go on forever, I return every sack given to me. If $n$th sack confers right to spend $n$th day in heaven, leads to hell-forever paradox.

In both stories, make infinitely many good trades and end up with less than I started with. “Paradox” is existence of 2-to-1 map from (smaller set) \{2, 3, ...\} to (bigger set) \{1, 2, ...\}.
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Money pile paradox

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Banker proposes to transfer a fraction (say 2/3) of each pile to the pile on its left and remainder to the pile on its right. Do this simultaneously for all piles.
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Every pile is bigger after transfer (and this can be true even if banker takes a portion of each pile as a fee).
You have an infinite collection of money piles with labels 0, 1, 2, … from left to right.

Precise details not important, but let’s say you have $5^n$ in the $n$th pile. Important thing is that pile size is increasing exponentially in $n$.

Banker proposes to transfer a fraction (say 2/3) of each pile to the pile on its left and remainder to the pile on its right. Do this simultaneously for all piles.

Every pile is bigger after transfer (and this can be true even if banker takes a portion of each pile as a fee).

Banker seemed to make you richer (every pile got bigger) but really just reshuffled your infinite wealth.
Two envelope paradox

- $X$ is geometric with parameter $1/2$. One envelope has $10^X$ dollars, one has $10^{X-1}$ dollars. Envelopes shuffled.
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- However, “Higher expectation given amount in first envelope” may not be right notion of “better.” If $S$ is payout with switching, $T$ is payout without switching, then $S$ has same law as $T - 1$. In that sense $S$ is worse.
Two envelope paradox

VALUE OF ENVELOPE ONE

- ($10000 with prob. 3/64) $468.75
- ($1000 with prob. 3/32) $18.75
- ($100 with prob. 3/16) $18.75
- ($10 with prob. 3/8) $3.75
- ($1 with prob. 1/4) $.25

VALUE OF ENVELOPE TWO

- ($12.50)
- ($6.25)
- ($2.50)
- ($1.25)
- ($.25)

($10000 with prob. 3/64) $468.75
($1000 with prob. 3/32) $18.75
($100 with prob. 3/16) $18.75
($10 with prob. 3/8) $3.75
($1 with prob. 1/4) $.25
Moral

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▶ Beware unbounded utility functions.
▶ They can lead to strange conclusions, sometimes related to “reshuffling infinite (actual or expected) wealth to create more” paradoxes.
▶ Paradoxes can arise even when total transaction is finite with probability one (as in envelope problem).