18.600: Lecture 8

Discrete random variables

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Outline

Defining random variables

Probability mass function and distribution function

Recursions

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- Question: What is $P\{X = k\}$ in this case?
- Answer: $\binom{n}{k}/2^n$, if $k \in \{0, 1, 2, \ldots, n\}$.

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- Does pairwise independence imply independence?
- No. Consider these three events: first coin heads, second coin heads, odd number heads. Pairwise independent, not independent.

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- $6/216$

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- Writing random variable as sum of indicators: frequently useful, sometimes confusing.

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- Are there other choices of S and P — and other functions X from S to P — for which the values of $P\{X = k\}$ are the same?
- Yes. "X is a Poisson random variable with intensity λ" is statement only about the *probability mass function* of X.

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- Famous correspondence by Fermat and Pascal. Led Pascal to write *Le Triangle Arithmétique*.