18.600: Lecture 3
What is probability?

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Outline

Formalizing probability

Sample space

DeMorgan's laws

Axioms of probability

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- **Market preference ("risk neutral probability"):** The market price of a contract that pays 100 if it rains tomorrow agrees with the price of a contract that pays 30 tomorrow no matter what.
- **Personal belief:** If you offered *me* a choice of these contracts, I'd be indifferent. (If need for money is different in two scenarios, I can replace dollars with "units of utility.")

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- Randomly throw a dart at a board. Sample space is the set of points on the board.

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- Denote by $\emptyset$ the set with no elements.

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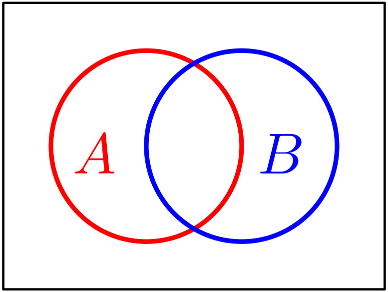
Venn diagrams



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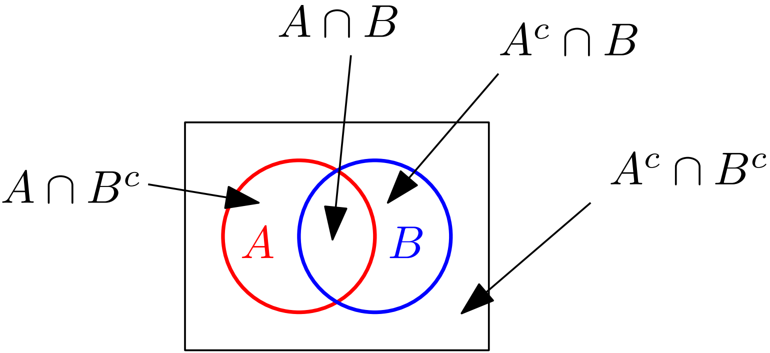

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- Countable additivity: $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ if $E_i \cap E_j = \emptyset$ for each pair i and j.

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- **Personal belief:** $P(A)$ is amount such that I'd be indifferent between contract paying 1 if A occurs and contract paying $P(A)$ no matter what. Seems to satisfy axioms with some notion of utility units, strong assumption of "rationality"...