18.600: Lecture 20

More continuous random variables

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- It is fun to learn their properties, symmetries, and interpretations.
- Today we'll discuss three of them that are particularly elegant and come with nice stories: Gamma distribution, Cauchy distribution, Beta bistribution.

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- So $\Gamma(\alpha)$ extends the function $(\alpha - 1)!$ (as defined for *strictly positive* integers α) to the positive reals.
- Vexing notational issue: why define Γ so that $\Gamma(\alpha) = (\alpha - 1)!$ instead of $\Gamma(\alpha) = \alpha!$?
- At least it's kind of convenient that Γ is defined on $(0, \infty)$ instead of $(-1, \infty)$.

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- Answer: $\binom{k-1}{n-1} p^{n-1}(1-p)^{k-n} p$.
- What's the continuous (Poisson point process) version of "waiting for the nth event"?

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- Write $p = \lambda/N$ and $k = xN$. (Note $p = \lambda x/k$.)
- For large N, $\binom{k-1}{n-1} p^{n-1}(1-p)^{k-n}p$ is

$$\frac{(k-1)(k-2)\dots(k-n+1)}{(n-1)!}p^{n-1}(1-p)^{k-n}p$$

$$\approx \frac{k^{n-1}}{(n-1)!}p^{n-1}e^{-x\lambda}p = \frac{1}{N}\Big(\frac{(\lambda x)^{(n-1)}e^{-\lambda x}\lambda}{(n-1)!}\Big).$$

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- Say that random variable X has gamma distribution with parameters (α, λ) if $f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1}e^{-\lambda x}\lambda}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

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- The general λ case is obtained by rescaling the $\lambda = 1$ case.

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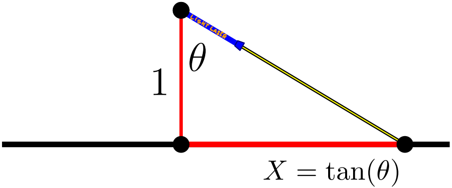

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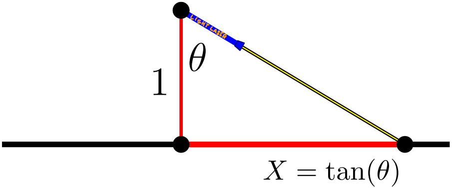

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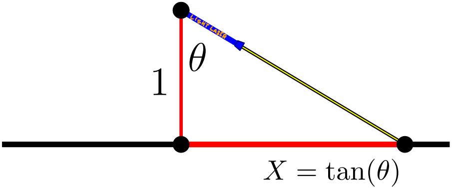

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- Find $f_X(x) = \frac{d}{dx}F(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

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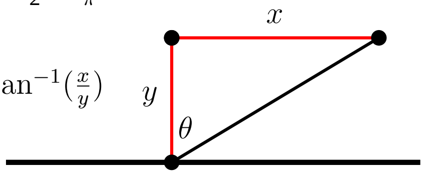
- FACT: start Brownian motion (x, y) in upper half plane. Probability it hits positive x-axis before negative x-axis is $\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(\frac{x}{y}) = \frac{1}{2} + \frac{1}{\pi}\theta$. Affine function of θ.

$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$



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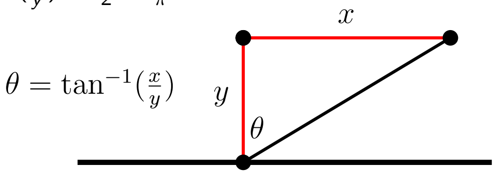

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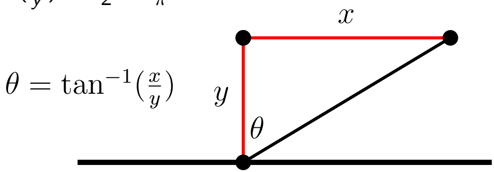

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- Applying FACT, translation invariance, reflection symmetry: $P\{X \leq x\} = P\{X \geq -x\} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$. So X is Cauchy.

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- Cauchy distribution doesn't have finite variance or mean.
- Some standard facts we'll learn later in the course (central limit theorem, law of large numbers) don't apply to it.

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- Given that number h of heads is $a-1$, and $b-1$ tails, what's *conditional* probability p was a certain value x?
- $P\Big(p = x | h = (a-1)\Big) = \frac{\frac{1}{11}\binom{n}{a-1}x^{a-1}(1-x)^{b-1}}{P\{h=(a-1)\}}$ which is $x^{a-1}(1-x)^{b-1}$ times a constant that doesn't depend on x.

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- What is $E[X]$?
- Answer: $\frac{a}{a+b}$.