18.600 Midterm 2, Spring 2019: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (10 points) Ramona enters a basketball free throw shooting contest and takes 100 shots. She makes each shot independently with probability .8 and misses with probability .2 Let X be the number of shots she makes.

(a) Compute the expectation and variance of X.

(b) Use a normal random variable to estimate the probability that she makes between 76 and 84 shots total. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer.

2. (20 points) Becky's Bagel Bakery does a brisk business. Customers arrive at random times, and each customer immediately purchases one type of bagel. The times $C_1, C_2, \ldots$ at which cinnamon raisin bagels are sold form a Poisson point process with a rate of 1 per minute. The times $P_1, P_2, \ldots$ at which pumpernickel bagels are sold form an independent Poisson point process with rate 2 per minute. And the times $E_1, E_2, \ldots$ at which everything bagels are sold form a Poisson point process with rate 3 per minute. Compute the following:

(a) The probability density function for C_3.

(b) The probability density function for $X = \min\{C_1, P_1, E_1\}$.

(c) The probability that *exactly* 10 bagels (altogether) are sold during the first 2 minutes the bakery is open.

(d) The expectation of $\cos(P_1 + E_1^2)$. (You can leave this as a double integral — no need to evaluate it.)

3. (10 points) Suppose that the pair of real random variables X, Y has joint density function $f(x, y) = \frac{1}{\pi^2(1+x^2)(1+y^2)}$.

(a) Compute the probability density function for $\frac{X+Y}{2}$.

(b) Compute the probability $P(X > Y + 2)$

4. (20 points) Suppose that X_1, X_2, X_3, X_4 are independent random variables, each of which has density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Compute the following:

(a) The correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

(b) The probability that $\min\{X_1, X_2\} > \max\{X_3, X_4\}$.

(c) The probability density function for $X_1 + X_2 + X_3$.

(d) The probability $P(X_1^2 + X_3^2 \leq 2)$. Give an explicit value.

5. (10 points) Imagine that A, B, C and D are independent uniform random variables on $[0, 1]$. You then find out that A is the third largest of those random variables.

(a) Given this new information, give a revised probability density function f_A for A (i.e., a Bayesian posteriori). **NOTE:** If you remember what this means, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$.

(b) According to your Bayesian prior, the expected value of A was $1/2$. *Given* that A was the third largest of the random variables, what is your *revised expectation* of the value A?

6. (15 points) Suppose that the pair (X, Y) is uniformly distributed on the triangle $T = \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}$. That is, the joint density function is given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & (x, y) \in T \\ 0 & (x, y) \notin T \end{cases}.$$

(a) Compute the marginal density function f_X.

(b) Compute the probability $P(X < 2Y)$.

(c) Compute the conditional density function $f_{X|Y=.5}(x)$.

7. (15 points) Suppose that X is an exponential random variable with parameter 1 and set $Z = X^5$.

(a) Compute the cumulative distribution function $F_Z(a)$ in terms of a.

(b) Compute the expectation $E[Z^2]$.

(c) Compute the conditional probability $P(Z > 32 | Z > 1)$.