18.600 Midterm 2, Spring 2018: 50 minutes, 100 points

1. **Carefully and clearly *show your work* on each problem (without writing anything that is technically not true). In particular, if you use any known facts (or facts proved in lecture) you should state clearly what fact you are using and why it applies.**

2. **No calculators, books, or notes may be used.**

3. **Simplify your answers as much as possible (but answers may include factorials — no need to multiply them out).**

1. (20 points) Suppose that X is a random variable with probability density function given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x/2 & 0 < x \leq 2 \\ 0 & x > 2 \end{cases}.$$

Suppose that Y is an independent random variable with the same probability density function. Write $Z = X^2 + Y^2$.

(a) Compute the joint density function $f_{X,Y}(x, y)$.

(b) Compute $E[Z]$. (You should be able to get an explicit number.)

(c) Compute $P(\max\{X, Y\} \leq 1)$.

2. (10 points) In a certain population, there are $n = 110000$ healthy people, each of whom has a $p = .01$ chance (independently of everyone else) of developing a certain disease during the course of a given decade. Let X be the number of people who develop the disease.

(a) Compute $E[X]$ and $\text{Var}[X]$.

(b) Use a normal random variable to estimate $P(1100 < X < 1133)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer.

3. (20 points) Suppose that X_1, X_2 and X_3 are independent random variables, each of which has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

Write $X = X_1 + X_2 + X_3$. Write $A = \min\{X_1, X_2, X_3\}$. Write $B = \max\{X_1, X_2, X_3\}$.

(a) Give a probability density function for X.

(b) Give a probability density function for A.

(c) Compute $E[B]$ and $\text{Var}[B]$.

4. (10 points) Suppose that X, Y, and Z are independent random variables, each of which has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$. Write $V = 3X$ and $W = X + Y + Z$.

(a) Compute the probability density function for V.

(b) Compute the probability density function for W.

5.(10 points) Let X and Y be independent standard normal random variables (so each has mean zero and variance one).

(a) Compute $P(X^2 + Y^2 \leq 1)$. Give an explicit value.

(b) Compute $P(\max\{|X|, |Y|\} \leq 1)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer.

6. (20 points) Let X and Y be independent uniform random variables on $[0, 1]$ and write $Z = X + Y$.

(a) Compute the conditional expectation $E[X|Z]$. (That is, express the random variable $E[X|Z]$ as a function of the random variable Z.)

(b) Compute the conditional expectation $E[Z|X]$. (That is, express the random variable $E[Z|X]$ as a function of the random variable X.)

(c) Compute the conditional variance $\mathrm{Var}[Z|Y]$. (That is, express the random variable $\mathrm{Var}[Z|Y]$ as a function of the random variable Y.)

(d) Compute the correlation coefficient $\rho(X, Z)$.

7. (10 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on the interval $[0, 1]$. Write $X = X_1 + X_2 + \ldots + X_n$.

(a) Compute the characteristic function $\phi_{X_1}(t)$.

(b) Compute the characteristic function $\phi_X(t)$.