Spring 2014 18.440 Final Exam Solutions

1. (10 points) Let X be a uniformly distributed random variable on $[-1, 1]$.

(a) Compute the variance of X^2. **ANSWER:**

$$\text{Var}(X^2) = E[(X^2)^2] - E[X^2]^2,$$

and

$$E[X^2] = \int_{-1}^{1} (x^2/2)dx = \frac{x^3}{6}\Big|_{-1}^{1} = 1/3,$$

$$E[(X^2)^2] = E[X^4] = \int_{-1}^{1} \frac{x^4}{2} dx = \frac{x^5}{10}\Big|_{-1}^{1} = 1/5,$$

so $\text{Var}(X^2) = E[(X^2)^2] - E[X^2]^2 = 1/5 - (1/3)^2 = 1/5 - 1/9 = 4/45$.

(b) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z? **ANSWER:** $F_{X_1}(a) = (a + 1)/2$ for $a \in [-1, 1]$. Thus

$$F_Z(a) = F_{X_1}(a)F_{X_2}(a) \ldots F_{X_n}(a) = \begin{cases} \left(\frac{a+1}{2}\right)^n & a \in [-1, 1] \\ 0 & a < -1 \\ 1 & a > 1 \end{cases}$$

2. (10 points) A certain bench at a popular park can hold up to two people. People in this park walk in pairs or alone, but nobody ever sits down next to a stranger. They are just not friendly in that particular way. Individuals or pairs who sit on a bench stay for at least 1 minute, and tend to stay for 4 minutes on average. Transition probabilities are as follows:

(i) If the bench is empty, then by the next minute it has a 1/2 chance of being empty, a 1/4 chance of being occupied by 1 person, and a 1/4 chance of being occupied by 2 people.

(ii) If it has 1 person, then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 1 person.

(iii) If it has 2 people then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 2 people.

(a) Use E, S, D to denote respectively the states empty, singly occupied, and doubly occupied. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by E, S, and D. **ANSWER:**

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix}$$

(b) If the bench is empty, what is the probability it will be empty two minutes later? **ANSWER:** $\frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{4} = 6/16 = 3/8$.

(c) Over the long term, what fraction of the time does the bench spend in each of the three states? **ANSWER:** We know

$$\begin{pmatrix} E & S & D \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix} = \begin{pmatrix} E & S & D \end{pmatrix}$$

and $E + S + D = 1$. Solving gives $E = S = D = 1/3$.

3. (10 points) Eight people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 8! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N]$ **ANSWER:** $8 \times \frac{1}{8} = 1$

(b) $P(N = 7)$ **ANSWER:** 0 since if seven get their own hat, then the eighth must also.

(c) $P(N = 0)$ **ANSWER:** This is an inclusion exclusion problem. Let A_i be the event that the ith person gets own hat. Then

$$\begin{aligned} P(N > 0) &= P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_8) \\ &= \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots \\ &= \binom{8}{1}\frac{1}{8} - \binom{8}{2}\frac{1}{8 \cdot 7} + \binom{8}{3}\frac{1}{8 \cdot 7 \cdot 6} \cdots \\ &= 1/1! - 1/2! + 1/3! + \ldots - 1/8! \end{aligned}$$

Thus,

$$P(N = 0) = 1 - P(N > 0) = 1 - 1/1! + 1/2! - 1/3! + 1/4! + 1/5! - 1/6! + 1/7! - 1/8! \approx 1/e.$$

4. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 5 with probability 1/2 and -5 with probability 1/2. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

(a) What is the probability that Y_n reaches 65 before the first time that it reaches -15? **ANSWER:** Y_n is a martingale, so by the optional stopping theorem, we have $E[Y_T] = Y_0 = 1$ (where $T = \min\{n : Y_n \in \{-15, 65\}\}$). We thus find $0 = Y_0 = E[Y_T] = 65p + (-15)(1-p)$ so $80p = 15$ and $p = 3/16$.

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = 5X_n$
(ii) $Z_n = 5^{-n} \prod_{i=1}^n X_i$
(iii) $Z_n = \prod_{i=1}^n X_i^2$
(iv) $Z_n = 17$
(v) $Z_n = X_n - 4$

ANSWER: (iv) only.

5. (10 points) Suppose that X and Y are independent exponential random variables with parameter $\lambda = 2$. Write $Z = \min\{X, Y\}$

(a) Compute the probability density function for Z. **ANSWER:** Z is exponential with parameter $\lambda + \lambda = 4$ so $F_Z(t) = 4e^{-4t}$ for $t \geq 0$.

(b) Express $E[\cos(X^2Y^3)]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:** $\int_0^\infty \int_0^\infty \cos(x^2y^3) \cdot 2e^{-2x} \cdot 2e^{-2y} dydx$

6. (10 points) Let X_1, X_2, X_3 be independent standard die rolls (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely). Write $Z = X_1 + X_2 + X_3$.

(a) Compute the conditional probability $P[X_1 = 6 | Z = 16]$. **ANSWER:** One can enumerate the six possibilities that add up to 16. These are $(4, 6, 6), (6, 4, 6), (6, 6, 4)$ and $(6, 5, 5), (5, 6, 5), (5, 5, 6)$. Of these, three have $X_1 = 6$, so $P[X_1 = 6 | Z = 16] = 1/2$.

(b) Compute the conditional expectation $E[X_2 | Z]$ as a function of Z (for $Z \in \{3, 4, 5, \ldots, 18\}$). **ANSWER:** Note that $E[X_1 + X_2 + X_3 | Z] = E[Z | Z] = Z$. So by symmetry and additivity of conditional expectation we find $E[X_2 | Z] = Z/3$.

7. (10 points) Suppose that X_i are i.i.d. uniform random variables on $[0,1]$.

(a) Compute the moment generating function for X_1. **ANSWER:** $E(e^{tX_1}) = \int_0^1 e^{tx}dx = \frac{e^t-1}{t}$.

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. **ANSWER:** $\left(\frac{e^t-1}{t}\right)^n$

8. (10 points) Let X be a normal random variable with mean 0 and variance 5.

(a) Compute $E[e^X]$. **ANSWER:** $E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{5}\sqrt{2\pi}} e^{-x^2/(2\cdot 5)} e^{-x} dx$. A complete the square trick allows one to evaluate this and obtain $e^{5/2}$.

(b) Compute $E[X^9 + X^3 - 50X + 7]$. **ANSWER:** $E[X^9] = E[X^3] = E[X] = 0$ by symmetry, so $E[X^9 + X^3 - 50X + 7] = 7$.

9. (10 points) Let X and Y be independent random variables. Suppose X takes values $\{1,2\}$ each with probability $1/2$ and Y takes values $\{1,2,3\}$ each with probability $1/3$. Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $H(X) = -(1/2)\log\frac{1}{2} - (1/2)\log\frac{1}{2} = -\log\frac{1}{2} = \log 2$. Similarly, $H(Y) = -(1/3)\log\frac{1}{3} - (1/3)\log\frac{1}{3} - (1/3)\log\frac{1}{3} = -\log\frac{1}{3} = \log 3$.

(b) Compute $H(X,Z)$. **ANSWER:** $H(X,Z) = H(X,Y) = H(X) + H(Y) = \log 6$.

(c) Compute $H(2^X 3^Y)$. **ANSWER:** Also $\log 6$, since each distinct (X,Y) pair gives a distinct number for $2^X 3^Y$.

10. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability $1/3$ and $.5$ with probability $2/3$. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^n X_i$ for $n \geq 1$.

(a) What is the the probability that Y_n reaches 4 before the first time that it reaches $\frac{1}{64}$? **ANSWER:** Y_n is a martingale, so by the optional stopping theorem, $E[Y_T] = Y_0 = 1$ (where $T = \min\{n : Y_n \in \{1/64, 4\}\}$). Thus $E[Y_T] = 4p + (1/64)(1-p) = 1$. Solving yields $p = 63/255 = 21/85$.

(b) Find the mean and variance of $\log Y_{400}$. **ANSWER:** $\log X_1$ is $\log 2$ with probability 1/3 and $-\log 2$ with probability 2/3. So

$$E[\log X_1] = \frac{1}{3}\log 2 + \frac{2}{3}(-\log 2) = \frac{-\log 2}{3}.$$

Similarly,

$$E[(\log X_1)^2] = \frac{1}{3}(\log 2)^2 + \frac{2}{3}(-\log 2)^2 = (\log 2)^2.$$

Thus,

$$\mathrm{Var}(X_1) = E[(\log X_i)^2] - E[\log X_i]^2 = (\log 2)^2 - \left(\frac{-\log 2}{3}\right)^2 = (\log 2)^2(1 - \frac{1}{9}) = \frac{8}{9}(\log 2)^2.$$

Multiplying, we find $E[\log Y_{400}] = 400E[\log X_1] = -400(\log 2)/3$.
And $\mathrm{Var}[\log Y_{400}] = (3200/9)(\log 2)^2$.

(c) Compute $\mathbb{E}Y_{100}$. **ANSWER:** Since Y_n is a martingale, we have $E[Y_{100}] = 1$. This can also be derived by noting that for independent random variables, the expectation of a product is the product of the expectations.