Martingales, risk neutral probability, and Black-Scholes option pricing

Supplementary notes for 18.600

These notes are adapted from the lecture slides used for Course 18.600 at MIT. We will cover the same material as the slides but with a few more words of explanation and illustration.

1 Defining martingales

Let S be a sample space. Let $X_0, X_1, X_2, \dots$ be a sequence of random variables. Informally, we will imagine that we are acquiring information about S in a sequence of stages, and the random variable X_n is a quantity that is known to us at the nth stage. If Z is any random variable, let

$$E[Z|\mathcal{F}_n]$$

denote the conditional expectation of Z *given* all the information that is available to us on the nth stage.[1] In other words, if you saw all the information you could obtain by stage n, and you made a Bayesian update to your probability distribution on S in light of this information, then $E[Z|\mathcal{F}_n]$ would represent the expected value of Z with respect to this revised probability. This definition may seem confusing in the abstract, but it should become clearer as we work through some examples. In practice it is often pretty straightforward to say, "Okay, if I were in the shoes of somebody who had all of the information available at stage n, what would I expect Z to be?" and to come up with an answer. This answer is $E[Z|\mathcal{F}_n]$.

If we don't specify otherwise, we assume that the information available at stage n consists precisely of the values $X_0, X_1, \dots, X_n$, so that

$$E[Z|\mathcal{F}_n] = E[Z|X_0, X_1, \dots, X_n].$$

However in some applications, one could imagine there are other things known as well at stage n. For example, maybe X_n represents the price of an asset X on the nth day and Y_n represents

[1] For the purposes of this course, it is enough for the reader to understand that $E[Z|\mathcal{F}_n]$ denotes the conditional expectation of Z *given* all the information that is available to us on the nth stage. However, in this footnote we briefly describe where this notation comes from and how it would be presented in more advanced treatments of this topic. The symbol $\mathcal{F}_n$ refers to a collection of subsets of S, which we interpret as the collection of all *events* A (recall that a subset of S is called an event) such that we can determine whether A occurs using only the information available at stage n. This $\mathcal{F}_n$ is a **σ-algebra**, which means that any finite or countable union of elements of $\mathcal{F}_n$ is again in $\mathcal{F}_n$, and that the complement of a set in $\mathcal{F}_n$ is again in $\mathcal{F}_n$. We assume $\mathcal{F}_0, \mathcal{F}_1, \dots$ is *increasing*, i.e., that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \dots$ because any yes-or-no question that can be answered at one stage can also be answered at any later stage (when one has even more information). An increasing sequence of σ-algebras is called a **filtration**. The X_n are assumed to be **adapted** with respect to the filtration, which essentially means that for any yes-no question one can ask about X_n, the event that the answer is yes is an element of $\mathcal{F}_n$. Less formally, X_n is adapted if the value X_n can be determined using the information available at stage n. For the purpose of this course, we will try not to get too bogged down in thinking about filtrations and σ-algebras. But these are things you are likely to see if you ever take a graduate probability course or read an academic paper about probability.

the price of asset Y on the nth day, and on day n one has access to the sequence $X_0, X_1, \dots, X_n$ *and* the sequence $Y_0, Y_1, \dots, Y_n$. Then $E[Z|\mathcal{F}_n]$ would be our revised expectation of Z after we have incorporated what we know about both sequences (up to stage n).

We say the X_n sequence is a **martingale** if $E[|X_n|] < \infty$ for all n and $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all n. Informally $X_0, X_1, \dots$ is a martingale if the following is true: taking into account all the information I have at stage n, the conditional expected value of X_{n+1} is just X_n.

To motivate this definition, imagine that X_n represents the price of a stock on day n. In this context, the martingale condition states informally that "The expected value of the stock tomorrow, given all I know today, is the value of the stock today." It is not too unreasonable to argue that stock prices should *approximately* have this property (on the scale of a single day) assuming we have no inside information and no dividends are being paid today or tomorrow. After all, if the stock price today were 50 and I expected it to be 60 tomorrow, then I would have an easy way to make money in expectation (buy today, sell tomorrow). But if the public had the same information I had, then other investors would also try to cash in on this by buying the stock today at 50, and people holding the stock would be reluctant to sell for 50. Indeed, we'd expect the price to be quickly bid up to about 60 *today.* A slightly more nuanced discussion of the applicability of martingales to finance (incorporating a few caveats) appears in the section on risk neutral probability below.

Let us now consider some simple examples. Suppose that $A_0, A_1, A_2, \dots$ are i.i.d. random variables each equal to -1 with probability .5 and 1 with probability .5. Let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n > 0$. Is the X_n sequence a martingale?

The answer is yes. To see this, note that

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n + A_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[A_{n+1}|\mathcal{F}_n]$$

by additivity of (conditional) expectation. Since X_n is known at stage n, we have $E[X_n|\mathcal{F}_n] = X_n$. Since we know nothing more about A_{n+1} at stage n than we originally knew, we have $E[A_{n+1}|\mathcal{F}_n] = 0$. Thus $E[X_{n+1}|\mathcal{F}_n] = X_n$ for all $n \geq 0$, so the sequence $X_0, X_1, \dots$ is indeed a martingale.

More informally, I'm just tossing a new fair coin at each stage to see if X_n goes up or down one step. If I know the information available up to stage n, and I know $X_n = 10$, then given everything I know, I see $X_{n+1} = 11$ and $X_{n+1} = 9$ as each having probability 1/2, so of course $E[X_{n+1}|\mathcal{F}_n] = 10 = X_n$.

To give another example, suppose each A_i is 1.01 with probability .5 and .99 with probability .5 and we write $X_0 = 1$ and $X_n = \prod_{i=1}^n A_i$ for $n > 0$? Then is X_n a martingale?

Again, the answer is yes. Note that $E[X_{n+1}|\mathcal{F}_n] = E[A_{n+1}X_n|\mathcal{F}_n]$. At stage n, the value X_n is known, and hence can be treated as a known constant, which can be factored out of the expectation, i.e., $E[A_{n+1}X_n|\mathcal{F}_n] = X_n E[A_{n+1}|\mathcal{F}_n]$. Since I know nothing new about A_{n+1} at stage n, we have $E[A_{n+1}|\mathcal{F}_n] = E[A_{n+1}] = 1$. Hence $E[A_{n+1}X_n|\mathcal{F}_n] = X_n$ for all $n \geq 0$, so the sequence $X_0, X_1, \ldots$ is indeed a martingale.

Stated informally, in this example I'm just tossing a new fair coin at each stage to see if X_n goes up or down by a percentage point of its current value. If I know all the information available up to stage n, and I know $X_n = c$, then I see $X_{n+1} = 1.01c$ and $X_{n+1} = .99c$ as equally likely, so $E[X_{n+1}|\mathcal{F}_n] = c = X_n$.

The above examples illustrate two important kinds of martingales: those obtained as sums of independent random variables (each with mean zero) and those obtained products of independent random variables (each with mean one). Note that in the two examples above, the precise probability distributions of the A_n do not matter as long as the A_n are independent of each other and all have mean zero (in the first case, involving sums) or mean one (in the second case, involving products).

Let's think about a few more examples of sequences of the form $X_0, X_1, \ldots$ and decide whether they are martingales.

1. The sequence $X_n = n$ is not a martingale: in this case $E[X_{n+1}|\mathcal{F}_n] = n + 1 \neq n$ when $n \geq 0$

2. The constant, deterministic sequence $X_n = 7$ is a martingale: in this case $E[X_{n+1}|\mathcal{F}_n] = 7 = X_n$ for all $n \geq 0$.

3. Suppose $A_1, A_2, \ldots$ are independent random variables with mean zero and variance one and write $S_0 = 0$ and $S_n = \sum_{i=1}^n A_n$ for $n \geq 1$. Then the sequence S_n is a martingale.

4. More surprisingly, if S_n is as in the previous example then the sequence $X_n = S_n^2 - n$ is a martingale. Why? First note that $E[X_n] = E[S_n^2 - n] = 0 = X_0$. To see this, recall that $E[S_n] = 0$ so $E[S_n^2] = \mathrm{Var}[S_n] = \sum_{j=1}^n \mathrm{Var}[A_j] = n$ by the additivity of variance for sums of independent random variables.

   But let us be careful to state that *the fact that $E[X_n] = X_0$ for all $n > 0$ is not by itself enough to imply that X_n is a martingale*. In order to see whether the sequence is a martingale, we need to show that $E[X_{n+1}|\mathcal{F}_n] = X_n$. This requires us to put ourselves in the shoes of somebody who has all the information available up until stage n and to then work out what that somebody would consider the expectation of X_{n+1} to be. To this end, note that at time n, we know X_n and S_n, so a person with the information available at time n can treat X_n and S_n as known constants. The only new information that we get as time goes from n to $n + 1$ is that we see the value A_{n+1}. Since we know nothing about A_{n+1}, its conditional mean and variance (given what we know up to stage n) are the same as its original mean and variance. So

$$
\begin{aligned}
E[X_{n+1}|\mathcal{F}_n] &= E[S_{n+1}^2 - (n+1)|\mathcal{F}_n] \\
&= E[(S_n + A_{n+1})^2|\mathcal{F}_n] - (n+1) \\
&= E[S_n^2 + 2A_{n+1}S_n + A_{n+1}^2] - (n+1) \\
&= S_n^2 + 0 + 1 - (n+1) = S_n^2 - n = X_n
\end{aligned}
$$

A **stopping time** is a non-negative integer-valued random variable T such that for all n the event that $T = n$ depends only on the information available to us at time n.[2] We can think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$. Informally, the statement that T is a stopping time means that the decision to sell at time n depends only the information we have up to time n, not on (as yet unknown) future prices. Specifying a stopping time can be interpreted as specifying a strategy for deciding when to sell the asset.

For example, let $A_1, A_2, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$. Which of the following is a stopping time?

1. The smallest T for which $|X_T| = 50$
2. The smallest T for which $X_T \in \{-30, 100\}$
3. The smallest T for which $X_T = 17$.
4. The T at which the X_n sequence achieves the value 17 for the 9th time.
5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.

Answer: first four, not last two.

2 Optional stopping theorem

2.1 Theorem statements

Doob's optional stopping time theorem is contained in many basic texts on probability and martingales. (See, for example, Theorem 10.10 of *Probability with Martingales*, by David Williams, 1991 — or just google Doob's optional stopping theorem and peruse the online sources.) It essentially says that you can't make money (in expectation) by buying and selling an asset whose price is a martingale. Precisely, if you buy the asset at some time and adopt any

[2]More formally, T is a stopping time if for each $n \geq 0$ the event that $T = n$ is an element of $\mathcal{F}_n$.

strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid. In other words, if the market price is a martingale, you cannot make money in expectation by "timing the market."

In the theorem statements below, note that when we say a random sequence $X_0, X_1, \ldots$ is bounded, we mean that for some $C > 0$, we have that with probability one $|X_n| \leq C$ for all $n \geq 0$. When we say the stopping time T is bounded, we mean that for some $C > 0$ we have $T \leq C$ with probability one.

Doob's Optional Stopping Theorem (first version): Suppose that X_0 is a known constant, that $X_0, X_1, X_2, \ldots$ is a **bounded** martingale, and that T is a stopping time. Then $E[X_T] = X_0$.

Doob's Optional Stopping Theorem (second version): Suppose that X_0 is a known constant, that $X_0, X_1, X_2, \ldots$ is a martingale, and that T is a **bounded** stopping time. Then $E[X_T] = X_0$.

Without at least one of these boundedness assumptions, the theorem would not be true. For a counterexample, recall that if $X_0 = 0$ and X_n goes up or down by 1 at each time step (each with probability .5) then $X_0, X_1, \ldots$ is a martingale. If we let T be the first n for which $X_n = 100$, then it is not too hard to show that T is a finite number with probability one. (That is, with probability one X_n reaches T *eventually*.) But then X_T is always 100, which means that $E[X_T] = 100 \neq X_0$.

Note however that X_n might reach some extremely negative values before it ever comes up to 100. So if you are a person making repeated one dollar bets up until the stopping time, and X_n represents your wealth at time n, you may find that there is a practical limit to how far negative your wealth can go (since at some point the casino is no longer willing to lend you money) and you cannot actually just "keep playing until you get to 100" in practice. The same would hold if you adopted the classical "double or nothing" strategy in which, each time you lose, you double the size of your bet and bet again, repeating this until *eventually* (with probability one) you win a bet and recover what you lost. In practice, it's pretty reasonable to assume that there are upper and lower bounds to your wealth, so that the optional stopping theorem would indeed hold.

2.2 Optional stopping theorem proof sketches

Let us sketch a quick proof by induction of the second version of the optional stopping theorem. Our inductive hypothesis will be the statement that "$E[X_T] = X_0$ if T is any stopping time which is at most K with probability one." Then clearly this statement is true if K is zero. So for the induction to work, we need to show that if this statement is true for any fixed non-negative positive integer K, then it is also true for $K + 1$. To establish the latter (while assuming the former) suppose that T is at most $K + 1$ with probability one. Let S be the minimum of T and K. By our inductive hypothesis, we know that $E[X_S] = X_0$. So to show that $E[X_T] = X_0$ it suffices to show that $E[X_T - X_S] = 0$. Note that the only way $X_T - X_S$ can be non-zero is if $S = K$ and $T = K + 1$, in which case $X_T - X_S$ represents the amount of money we make on the $(K + 1)$th step. So we just need to show that the expected amount of money we make on the $(K + 1)$th step (if we haven't sold the stock by time K) is zero. To see this, recall that $X_0, X_1, \ldots$ is a martingale, which means that $E[X_{K+1} - X_K|\mathcal{F}_K] = 0$. This means that given *everything* we know up to time K (including our knowledge of whether or not we have already sold the stock at time K) we will always still expect $X_{K+1} - X_K$ to be zero. Since this is true for any possible scenario (of what the information looks like at time K) we may conclude by averaging over the possible scenarios that *overall* we have $E[X_T - X_S] = 0$. Hence $E[X_T] = E[X_S] = X_0$, which implies that our inductive hypothesis holds for $K + 1$ (and by induction for all positive integers).

The first version of the optional stopping theorem can be derived from the second version using a limiting procedure. The idea of the proof is that one lets T_K be the minimum of T and K and attempts to show that

$$\lim_{K\to\infty} E[X_{T_K}] = E[X_T].$$

The limit on the left hand side is obviously X_0 (since each term in the sequence is X_0, by the second version of the optional stopping theorem) so this would imply the desired conclusion: that $E[X_T] = X_0$.

To implement this strategy, recall that we are assuming that the $|X_{T_K}|$ are with probability one all bounded by a fixed constant $C > 0$. Since T_K is almost surely finite, it follows that for any $\epsilon > 0$ we may choose K large enough so that

$$P(T_K \neq T) = P(T_K > K) < \epsilon.$$

Since X_T and X_{T_K} only differ with probability at most ϵ (and the magnitude of that difference is always at most $2C$) it follows that

$$E[X_{T_K}] - E[X_T] = E[X_{T_K} - X_T] \in [2C\epsilon, -2C\epsilon].$$

Taking ϵ small enough, we can make this interval as small as we want. Since we know that $E[X_{T_K}] = X_0$ for all K, this can only be true if $E[X_T] = X_0$.[3]

3 More problems and perspectives

Here are a couple of martingale questions that can be solved with the optional stopping theorem.

[3] We remark that there are other variants of the optional stopping theorem (i.e., other sufficient conditions on the martingale and the stopping time that together ensure that $E[X_T] = X_0$) that we will not discuss here. For people inclined to look up these other versions, "uniform integrability" is one of the key phrases that comes up, as is "convergence in L^p."

1. Suppose that an asset price is a martingale $X_0, X_1, \dots$ that starts at $X_0 = 50$ and changes by increments of ± 1 at each time step. What is the probability that the price goes down to 40 before it goes up to 70? To answer this, let T be the first time n for which X_n is 40 or 70. Write $p_{40} = P(X_T = 40)$ and $p_{70} = P(X_T = 70)$. Then $E[X_T] = 40p_{40} + 70p_{70}$ is equal to $X_0 = 50$ by the optional stopping theorem. Since we also know $p_{40} + p_{70} = 1$ we can solve the two linear equations in two unknowns to get $p_{40} = 2/3$ and $p_{70} = 1/3$.

   Another way to solve this problem is to rescale so that the endpoints are zero or one. Write $Y_n = (X_n - 40)/30$. One can use linearity of expectation to show that an affine function of a martingale is also a martingale, so Y_n is also a martingale. But Y_0 starts at $1/3$ and we have Y_T equal to either 0 or 1. Since $E[Y_T] = 1/3$ we must have $P(Y_T = 1) = 1/3$ and $P(Y_T = 0) = 2/3$.

   Generally, this argument shows that if we have a bounded martingale starting at a point c between a and b and stopping when it hits a or b (assuming it reaches one or the other eventually with probability one), the probability it hits b first is $(c-a)/(b-a)$. In other words, if the martingale starts a p fraction of the way from a to b, then it will get to b (before getting back to a) with probability p.

2. What is the probability that the martingale from the previous example goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100? To answer this we just use the analysis from the last problem and multiply. First, we have a $10/11$ chance to get down to 45 (before hitting 100). Then, given that that succeeds, we have a $9/11$ chance to get to 55 (before hitting 0). Then a $9/11$ chance to get down to 45 again (before hitting 100) and a $9/11$ chance to get back to 55 again (before hitting 0). We end up with $(10/11)(9/11)^3 \approx .4979$.

Next, let us make a couple more observations. First, observe that the two-element sequence $E[X], X$ is clearly martingale. Second, recall that we have interpreted the conditional expectation $E[X|Y]$ as a random variable, which happens to depend only on the value of Y. It describes the expectation of X given observed Y value. Then observe the following.

1. $E[E[X|Y]] = E[X]$, which means that the three-element sequence $E[X], E[X|Y], X$ is a martingale.

2. More generally if Y_i are any random variables, the sequence

$$E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \dots$$

   is a martingale.

3. Stlll more generally, the sequence

$$E[X|\mathcal{F}_0], E[X|\mathcal{F}_1], E[X|\mathcal{F}_2], \dots$$

is a martingale if X is any fixed random variable, and we have a sample space that we are learning information about in stages.

For a story example, let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale. As another example, let A_i be my best guess at the probability that a basketball team will win the game, given the outcome of the first i minutes of the game. Then (assuming some "rationality" of my personal probabilities) A_i is a martingale.

4 Risk neutral probability and martingales

According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**.[4]

To explain, what this means, we recall that "Risk neutral probability" is a fancy term for "market probability". (The term "market probability" is arguably more descriptive.) That is, it is a probability measure that you can deduce by looking at prices on a market. For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made? If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75. Risk neutral probability is the probability determined by the market betting odds. More precisely:

Risk neutral probability[5] **of event** A**:** $P_{RN}(A)$ denotes

$$\frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs }\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what }\}}.$$

If the risk-free interest rate is constant and equal to r (compounded continuously), then the denominator is e^{-rT}. Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies the axioms of probability. That is, $0 \le P_{RN}(A) \le 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

Arbitrage example: here is an example of an arbitrage one can implement when one of the axioms of probability is violated. If A and B are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then

[4] The reader who is interested in the financial issues raised in this section (and who wants a more precise statement of this theorem) can read many more details about the subject in *Mathematics for Finance: An Introduction to Financial Engineering* by Zastawniak and Capiński. Google it.

[5] In these notes, we are assuming that there is a liquid market for the contracts we discuss and that the bid-ask spread is very small — so that contracts like this have a price such that it is possible to both buy and sell at (very close to) that price.

we sell contracts paying 1 if A occurs and 1 if B occurs, buy a contract paying 1 if $A \cup B$ occurs, and pocket the difference. No matter what the outcome is, the amount we end up owing will equal the amount that is owed to us, so (assuming we trust that all contracts will be honored and enforced[6]) we are taking on no risk.

Similar things can be done if the other axioms are violated. This is how one shows that the absence of arbitrage opportunities implies that the axioms apply.

At first sight, one might think that $P_{RN}(A)$ describes the market's best guess at the probability that A will occur. But suppose A is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$? It should be 0. Even if people think A is *likely*, a contract paying a dollar when A occurs is worthless. Now, suppose there are only 2 outcomes: A is the event that the economy booms and everyone prospers and B is the event that economy sags and everyone is needy. Suppose the purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

The answer is that we should expect $P_{RN}(A) < .5$. People are risk averse. In the second scenario they need the money more.

Suppose that A is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market's best assessment of) the probability that the Red Sox will win?

In this case the answer is arguably yes. The amount that *people in general* need or value dollars does not depend much on whether A occurs (even though the financial needs of specific individuals may depend heavily on A). Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to the team's prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of

[6]Of course, this assumption is not always justified in practice. In the runup to the 2012 presidential election, two large and liquid betting sites, Intrade and Betfair, offered very different odds for the same election: Intrade giving Romney a relatively higher chance of winning, Betfair giving Obama a relatively higher chance. It appeared that, for much less than \$100, one could buy two contracts: one on Intrade paying \$100 if Obama won, and one on Betfair paying \$100 if Obama didn't win. This was a classical arbitrage opportunity. Ordinarily, one would expect traders to try to take advantage of this opportunity, and one would expect the actions of these traders to cause the discepancy to go away quickly. In this case a major gap persisted for weeks. Speculation about the reasons for the persistent discrepancy appears for example here `http://www.overcomingbias.com/2012/11/was-intrade-being-manipulated-over-the-last-month.html`. Shortly following the election and Obama's victory, the US government announced that it was cracking down on Intrade, various financial problems within Intrade became apparent, and it became unclear whether Intrade would be able to redeem the money it owed its customers as the company collapsed. `http://business.time.com/2013/03/11/online-predictions-market-intrade-shuts-down-months-after-federal-lawsuit`. Given this history, one might be tempted to say, "Okay, maybe that's why the professional traders didn't take advantange of the arbitrage and close the gap. They knew that Intrade *might* be on the verge of collapse." But it's hard to know if this is actually what traders were thinking. The chance that the company managing and enforcing the contracts might collapse is sometimes called "third party risk." Collapses of this kind (involving institutions much larger and fundamental to our financial system than Intrade and Betfair) played a role in the 2008 financial crisis.

what well-informed informed experts would consider the true probability.

The definition of risk neutral probability depends on choice of currency (the so-called *numéraire*). In the 2016 US presidential election, investors predicted (correctly) that the value of the Mexican peso (in US dollars) just after the election would be substantially lower if Trump won. Although this example was not as extreme as the one mentioned above (where one candidate would declare one of the currencies to be worthless), it was still significant enough for the risk neutral probability of a Trump victory to be quite different depending on whether one used dollars or pesos as the numéraire

We remark that risk neutral probability can also be defined for variable times and variable interest rates — e.g., one can take the numéraire to be the amount one dollar in a variable-interest-rate money market account has grown to when the outcome is known. We can define $P_{RN}(A)$ to be the price of a contract paying this amount if and when A occurs. For simplicity, we focus on a fixed future time T and a fixed interest rate r in these notes.

By assumption, the price of a contract that pays one dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$. If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?

The answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$. More generally, in the absence of arbitrage, the price of a contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability. For example, if a non-divided paying stock will be worth X at time T, then its price today should be $E_{RN}(X)e^{-rT}$. As mentioned above, the so-called **fundamental theorem of asset pricing** states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. The current price of the stock being $E_{RN}(X)e^{-rT}$ follows from this.

5 Black-Scholes

Famous professors who worked at MIT at some point (Black, Scholes, and Merton) won the 1997 Nobel Prize for their work on an option pricing model now known as the Black-Scholes model. The mathematics of our Black-Scholes discussion will not go far beyond things we know. The main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices). We can interpret our analysis in this section as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, etc.) Much has been written about the Black-Scholes formula (start with the Wikipedia articles if you want to learn more). These notes will give a very quick overview and will explain how the formula can be derived directly from a few simple assumptions about risk neutral probability.

Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a *continuous time martingale*. The important thing to know about it for now is that the value of the Brownian motion at time T is a normal random variable with mean zero and variance $T\sigma^2$ where σ^2 is a **volatility** parameter. The Black-Scholes theory assumes that the log of an asset price is a process called *Brownian motion with drift* with respect to *risk neutral probability*. Since we will focus on a fixed future time T in these notes, the important thing about this assumption is that it implies that the log of the asset price at time T is a normal random variable with variance $T\sigma^2$ and *some* fixed mean value.

1. **Assumption:** the log of an asset price X at a fixed future time T is a normal random variable (call it N) with some known variance (call it $T\sigma^2$) and some mean (call it μ) with respect to risk neutral probability.

2. **Observation:** N normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

3. **Observation:** If X_0 is the current price then

$$X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}.$$

4. **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.[7]

5. **Conclusion:** If g is any function then the price of a contract that pays $g(X)$ at time T is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

   where N is normal with mean μ and variance $T\sigma^2$.

A **European call option** on a stock at **maturity date** T, **strike price** K, gives the holder the right (but not obligation) to purchase a share of stock for K dollars at time T.

> The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. $\mathcal{SS}$

[7]This is a very important point. Previous works on options pricing had assumed that one somehow had to know μ in advance to price options — one needed a guess about the direction the stock was drifting. The Black-Scholes work notes that the relevant notion of probability for determining prices is risk neutral probability, not some notion of "true probability," and that with respect to risk neutral probability the value of μ is *determined* by the values of r and σ, and hence is not needed as an input.

If X is the value of the stock at T, then the value of the option at time T is given by $g(X) = \max\{0, X - K\}$. The Black-Scholes formula states that the price of a contract paying $g(X)$ at time T is

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

where N is normal with variance $T\sigma^2$, mean $\mu = \log X_0 + (r - \sigma^2/2)T$.

We could just end the discussion here, but let's try to put this expression into a more explicit form. Write this as

$$e^{-rT}E_{RN}[\max\{0, e^N - K\}] = e^{-rT}E_{RN}[(e^N - K)1_{N \geq \log K}]$$

$$= e^{-rT}\int_{\log K}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx.$$

Recall that we let T be the time to maturity, the X_0 current price of underlying asset, K the strike price, r the risk free interest rate, and σ^2 the volatility. We need to compute $e^{-rT}\int_{\log K}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(x-\mu)^2}{2T\sigma^2}}(e^x - K)dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$. We can write this as

$$e^{-rT}\int_{\log K}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} e^x dx - e^{-rT}\int_{\log K}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} K dx.$$

We can use a complete-the-square trick to deal with the extra e^x in the first term. We can also us generally the fact that the probability a normal random variable is more than a standard deviations above its mean is given by $1 - \Phi(a)$ (which implies a statement about the integral of the density function from some point to infinity). These ideas allow us to compute the two terms explicitly in terms of the standard normal cumulative distribution function Φ. We leave the details as an exercise to the reader. In the end we find that the price of European call is

$$\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$$

where $d_1 = \frac{\ln(\frac{X_0}{K}) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(\frac{X_0}{K}) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$.

6 Call quotes and risk neutral probability

If $C(K)$ is the price of a European call with strike price K and $f = f_X$ is the risk neutral probability density function for X at time T, then $C(K) = e^{-rT}\int_{-\infty}^{\infty} f(x)\max\{0, x - K\}dx$. Differentiating under the integral, we find that

$$e^{rT}C'(K) = \int f(x)(-1_{x>K})dx = -P_{RN}\{X > K\} = F_X(K) - 1,$$

$$e^{rT}C''(K) = f(K).$$

We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time T at cboe.com and work out approximately what F_X and hence f_X must be.

Try doing this an option with a date in the near future, so that one can assume that e^{rT} is essentially one. You'll find when you look up the option chain that one is not given $C(K)$ for all values of K (it is only listed for a discrete set of K values) so one has to estimate what would be its first and second derivatives of C from this. Still it is satisfying to know that you can use this technique to assess the risk neutral probability that a stock price will lie in a specified range on a specified date. If you are ever offered a job at a company that promises to pay you in stock or in options, you might want to take a look at the option prices for the company and try to work out a probability distribution for the value of your pay.

The risk neutral probability densities derived from call quotes are not quite lognormal in practice. The tails are too fat. In other words, the risk neutral probability that the stock will rise or fall by very large factors tends to be higher than the Black-Scholes model would predict.

Although Black-Scholes is not a perfect predictor of option prices, traders still think about the model when they think about pricing. When looking at a specific option, the "implied volatility" is defined to be the value of σ^2 that (when plugged into Black-Scholes formula along with the other known parameters) predicts the current market price. If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the "volatility smile"), it is not constant. Nonetheless, comparing "implied volatilities" gives traders an intuitive way to understand option prices.

The main Black-Scholes assumption is that risk neutral probability densities are lognormal. The heuristic support for this assumption is basically this: if the price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Then the central limit theorem gives log normality for large T. However, in reality, the amount that a stock varies up and down can differ a lot from one day to another.

It is also the case in principle that prices can have big jumps. Although we will not discuss them here, we remark that there are variants of the Black-Scholes model that allow for variable volatility, random interest rates, processes with random jump discontinuities (called Lévy processes) and so forth.