18.600: Lecture 21

Joint distributions functions

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Outline

Distributions of functions of random variables

Joint distributions

Independent random variables

Examples

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- If $Z = X^2$, then what is $P\{Z \leq 16\}$?

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- Given the joint distribution of X and Y, we sometimes call distribution of X (ignoring Y) and distribution of Y (ignoring X) the **marginal** distributions.
- In general, when X and Y are jointly defined discrete random variables, we write $p(x, y) = p_{X,Y}(x, y) = P\{X = x, Y = y\}$.

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- Question: if I tell you the two parameter function F, can you use it to determine the marginals F_X and F_Y?
- Answer: Yes. $F_X(a) = \lim_{b \to \infty} F(a,b)$ and $F_Y(b) = \lim_{a \to \infty} F(a,b)$.

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- From this, we can show that it works for strips, rectangles, general open sets, etc.

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- When X and Y are continuous, they are independent if $f(x, y) = f_X(x)f_Y(y)$.

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- Using polar coordinates, we want $\int_0^1 (2\pi r)\frac{1}{2\pi}e^{-r^2/2}dr = -e^{-r^2/2}\Big|_0^1 = 1 - e^{-1/2} \approx .39$.

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- Can we get the marginals from that?

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- Are all of the T_i and A_i independent of each other? What are their probability distributions?

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- Can hiker breathe sigh of relief after 5 attack-free hours?

Buffon's needle problem

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- Draw the box $[0,1]\times[0,\pi]$ on which (X,θ) is uniform. What's the area of the subset where $X \ge 1-\sin\theta$?