18.175: Lecture 4

Expectation properties, law of large numbers statement, and Kolmogorov's extension theorem

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Outline

Lebesgue integration and expectation

Stating the law of large numbers

Kolmogorov extension theorem

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- Then extend to case $\mu(\Omega) = \infty$.

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- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{R}^d, \lambda)$, write $\int_E f(x) dx = \int 1_E f d\lambda$.

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- EX^k is called k**th moment of** X. Also, if $m = EX$ then $E(X - m)^2$ is called the **variance** of X.

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- **Jensen's inequality:** If μ is probability measure and $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(\int f d\mu) \leq \int \phi(f) d\mu$. If X is random variable then $E\phi(X) \geq \phi(EX)$.

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- **Cauchy-Schwarz inequality:** Special case $p = q = 2$. Gives $\int |fg| d\mu \leq \|f\|_2 \|g\|_2$. Says that dot product of two vectors is at most product of vector lengths.

Bounded convergence theorem

- **Bounded convergence theorem:** Consider *probability* measure μ and suppose $|f_n| \leq M$ a.s. for all n and some fixed $M > 0$, and that $f_n \to f$ in probability (i.e., $\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0$ for all $\epsilon > 0$). Then

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- **Main idea of proof:** for any ϵ, δ can take n large enough so $\int |f_n - f| d\mu < M\delta + \epsilon$.

Fatou's lemma

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$$\liminf_{n\to\infty} \int f_n d\mu \geq \int (\liminf_{n\to\infty} f_n) d\mu.$$

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(Counterexample for opposite-direction inequality using thin and tall rectangles?)
- **Main idea of proof:** first reduce to case that the f_n are increasing by writing $g_n(x) = \inf_{m\geq n} f_m(x)$ and observing that $g_n(x) \uparrow g(x) = \liminf_{n\to\infty} f_n(x)$. Then truncate, used bounded convergence, take limits.

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- **Main idea of proof:** Fatou for functions $g + f_n \geq 0$ gives one side. Fatou for $g - f_n \geq 0$ gives other.

Computing expectations

- Change of variables. Measure space $(\Omega, \mathcal{F}, P)$. Let X be random variable in $(S, \mathcal{S})$ with distribution μ. Then if $f(S, \mathcal{S}) \to (R, \mathcal{R})$ is measurable we have $Ef(X) = \int_S f(y)\mu(dy)$.

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- Examples: normal, exponential, Bernoulli, Poisson, geometric...

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Strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \ldots$ are i.i.d. real-valued random variables with expectation m and $A_n := n^{-1} \sum_{i=1}^{n} X_i$ are the *empirical means* then $\lim_{n \to \infty} A_n = m$ almost surely.

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- What does i.i.d. mean?
- Answer: independent and identically distributed.
- Okay, but what does independent mean in this context? And how do you even define an infinite sequence of independent random variables? Is that even possible? It's kind of an empty theorem if it turns out that the hypotheses are never satisfied. And by the way, what measure space and σ-algebra are we using? And is the event that the limit exists even measurable in this σ-algebra? Because if it's not, what does it mean to say it has probability one? Also, why do they call it the strong law? Is there also a weak law?

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- Random variables X and Y are independent if for all $C, D \in \mathcal{R}$, we have $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$, i.e., the events $\{X \in C\}$ and $\{Y \in D\}$ are independent.

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- Two σ-fields $\mathcal{F}$ and $\mathcal{G}$ are independent if A and B are independent whenever $A \in \mathcal{F}$ and $B \in \mathcal{G}$. (This definition also makes sense if $\mathcal{F}$ and $\mathcal{G}$ are arbitrary algebras, semi-algebras, or other collections of measurable sets.)

Independence of multiple events/random variables/σ-algebras

- Say events $A_1, A_2, \ldots, A_n$ are independent if for each $I \subset \{1, 2, \ldots, n\}$ we have $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$.

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- Say σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ if any collection of events (one from each σ-algebra) are independent. (This definition also makes sense if the $\mathcal{F}_i$ are algebras, semi-algebras, or other collections of measurable sets.)

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Independence theorem

- **Theorem:** If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are independent, and each $\mathcal{A}_i$ is a π-system, then $\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_n)$ are independent.

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- **Main idea of proof:** Apply the π-λ theorem.

Kolmogorov's Extension Theorem

- **Task: make sense of this statement.** Let Ω be the set of all countable sequences $\omega = (\omega_1, \omega_2, \omega_3 \dots)$ of real numbers. Let $\mathcal{F}$ be the smallest σ-algebra that makes the maps $\omega \to \omega_i$ measurable. Let P be the probability measure that makes the ω_i independent identically distributed normals with mean zero, variance one.

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- The $\mathcal{F}$ described above is the natural product σ-algebra: smallest σ-algebra generated by the "finite dimensional rectangles" of form $\{\omega : \omega_i \in (a_i, b_i], 1 \leq i \leq n\}$.
- Question: what things are in this σ-algebra? How about the event that the ω_i converge to a limit?

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- Proved using semi-algebra variant of Carathéeodory's extension theorem.