18.175: Lecture 20

Markov chains

Scott Sheffield

MIT

Outline

Review what you know about finite state Markov chains

Finite state ergodicity and stationarity

More general setup

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- Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.

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- Sequence is called a **Markov chain** if we have a fixed collection of numbers P_{ij} (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state i, there is probability P_{ij} that system will next be in state j.

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- Kind of an "almost memoryless" property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).

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- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?
- Over the long haul, what fraction of days are sunny?

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- For this to make sense, we require $P_{ij} \geq 0$ for all i, j and $\sum_{j=0}^{M} P_{ij} = 1$ for each i. That is, the rows sum to one.

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- How about the following product?

$$\begin{pmatrix} p_0 & p_1 & \dots & p_M \end{pmatrix} A^n$$

- Answer: the probability distribution at time n.

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- If A is the one-step transition matrix, then A^n is the n-step transition matrix.

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- What if each P_{ij} is either one or zero?
- Answer: state evolution is deterministic.

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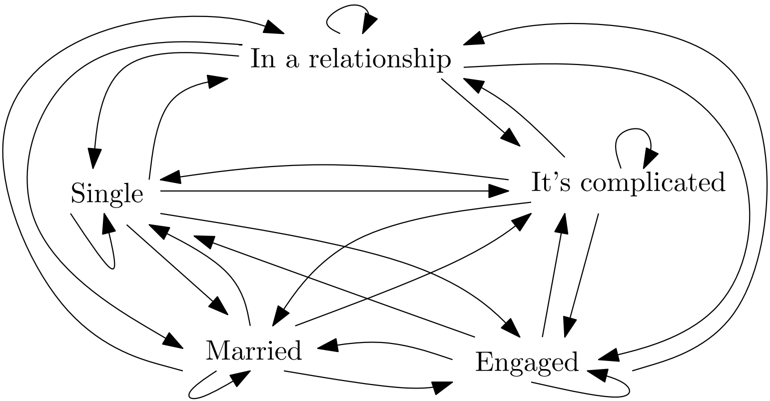
- Can compute $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix}$

Does relationship status have the Markov property?



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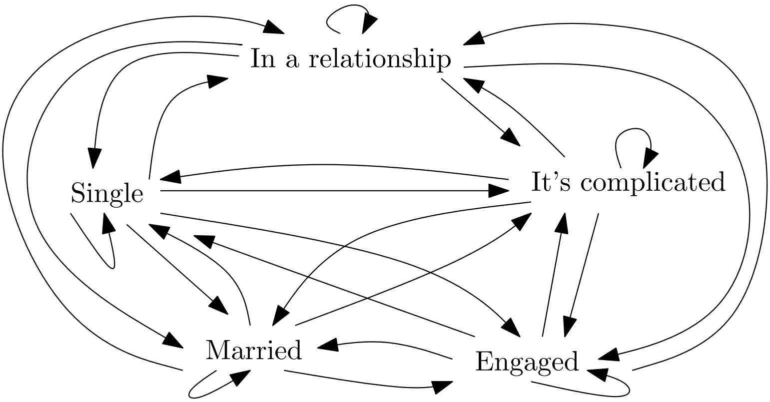

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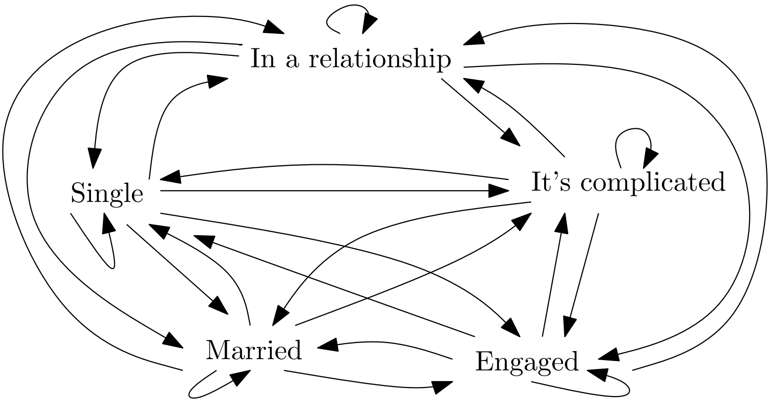

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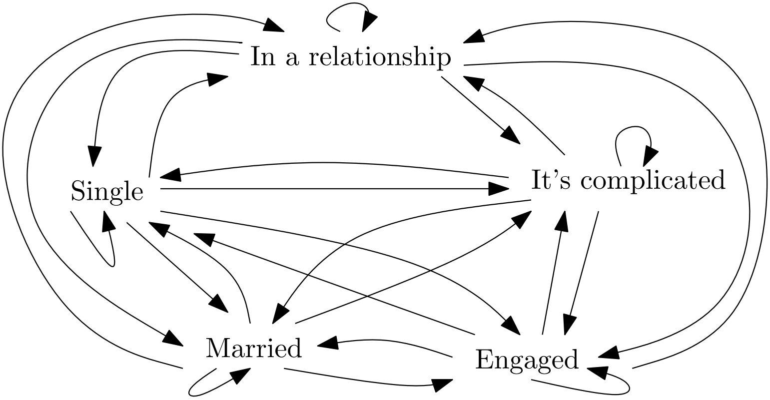

- Can we assign a probability to each arrow?
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- Not true... Can we make a better model with more states?

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- Turns out that if chain has this property, then $\pi_j := \lim_{n\to\infty} P_{ij}^{(n)}$ exists and the π_j are the unique non-negative solutions of $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ that sum to one.

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- One can solve the system of linear equations $\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$ to compute the values π_j. Equivalent to considering A fixed and solving $\pi A = \pi$. Or solving $(A - I)\pi = 0$. This determines π up to a multiplicative constant, and fact that $\sum \pi_j = 1$ determines the constant.

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- This means that $.5\pi_0 + .2\pi_1 = \pi_0$ and $.5\pi_0 + .8\pi_1 = \pi_1$ and we also know that $\pi_1 + \pi_2 = 1$. Solving these equations gives $\pi_0 = 2/7$ and $\pi_1 = 5/7$, so $\pi = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix}$.

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- Recall that $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}$

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- How do we construct an infinite Markov chain? Choose p and initial distribution μ on $(S, \mathcal{S})$. For each $n < \infty$ write

$$P(X_j \in B_j, 0 \leq j \leq n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots$$

$$\int_{B_n} p(x_{n-1}, dx_n).$$

Extend to $n = \infty$ by Kolmogorov's extension theorem.

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- **Notation:** Extension produces probability measure P_μ on sequence space $(S^{0,1,\ldots}, \mathcal{S}^{0,1,\ldots})$.
- **Theorem:** $(X_0, X_1, \ldots)$ chosen from P_μ is Markov chain.

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- **Definition, again:** Say X_n is a **Markov chain** w.r.t. $\mathcal{F}_n$ with transition probability p if $P(X_{n+1} \in B | \mathcal{F}_n) = p(X_n, B)$.
- **Construction, again:** Fix initial distribution μ on $(S, \mathcal{S})$. For each $n < \infty$ write

$$P(X_j \in B_j, 0 \leq j \leq n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots$$

$$\int_{B_n} p(x_{n-1}, dx_n).$$

  Extend to $n = \infty$ by Kolmogorov's extension theorem.
- **Notation:** Extension produces probability measure P_μ on sequence space $(S^{0,1,\ldots}, \mathcal{S}^{0,1,\ldots})$.
- **Theorem:** $(X_0, X_1, \ldots)$ chosen from P_μ is Markov chain.
- **Theorem:** If X_n is any Markov chain with initial distribution μ and transition p, then finite dim. probabilities are as above.

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- Ehrenfest chain.