18.175: Lecture 15

Random walks

Scott Sheffield

MIT

Outline

Risk neutral probability

Random walks

Stopping times

Arcsin law, other SRW stories

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- For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
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- If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
- Risk neutral probability is the probability determined by the market betting odds.

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event** A**:** $P_{RN}(A)$ denotes

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- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

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- **Arbitrage example:** if A and B are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if A occurs and 1 if B occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.

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- Now, suppose there are only 2 outcomes: A is event that economy booms and everyone prospers and B is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

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- Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.

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- Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team's prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.

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- Or, for simplicity, can focus on fixed time T, fixed interest rate r.

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- **Aside:** So-called **fundamental theorem of asset pricing** states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.

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- This is related to the tail σ-algebra we introduced earlier in the course. Bigger or smaller?

Hewitt-Savage 0-1 law

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- If $X_1, X_2, \ldots$ are i.i.d. and $A \in \mathcal{A}$ then $P(A) \in \{0, 1\}$.
- **Idea of proof:** Try to show A is independent of itself, i.e., that $P(A) = P(A \cap A) = P(A)P(A)$. Start with measure theoretic fact that we can approximate A by a set A_n in σ-algebra generated by $X_1, \ldots X_n$, so that symmetric difference of A and A_n has very small probability. Note that A_n is independent of event A_n' that A_n holds when $X_1, \ldots, X_n$ and $X_{n_1}, \ldots, X_{2n}$ are swapped. Symmetric difference between A and A_n' is also small, so A is independent of itself up to this small error. Then make error arbitrarily small.

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  - $S_n \to -\infty$
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- **Idea of proof:** Hewitt-Savage implies the $\limsup S_n$ and $\liminf S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if X_1 is not a.s. constant, then both values would depend on X_1 if they were not in $\pm\infty$

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- In finance applications, T might be the time one sells a stock. Then this states that the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

Stopping time examples

- Let $A_1, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$.

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- Let $A_1, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^n A_i$ for $n \geq 0$.
- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
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- Answer: first four, not last two.

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- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.

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- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?

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- What is $\mathbb{E}N$?

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Reflection principle

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- How many walks from $(0, x)$ to (n, y) that don't cross the horizontal axis?
- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

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- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

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- Theorem for amount of positive positive time.