18.175: Lecture 14

Infinite divisibility and so forth

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Outline

Infinite divisibility

Higher dimensional CFs and CLTs

Random walks

Stopping times

Arcsin law, other SRW stories

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- More general constructions are possible via Lévy Khintchine representation.

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- The inversion theorems and continuity theorems that apply here are essentially the same as in the one-dimensional case.

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- Let $\mathcal{E}$ be the σ-field of permutable events.
- This is related to the tail σ-algebra we introduced earlier in the course. Bigger or smaller?

Hewitt-Savage 0-1 law

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Hewitt-Savage 0-1 law

- If $X_1, X_2, \dots$ are i.i.d. and $A \in \mathcal{A}$ then $P(A) \in \{0, 1\}$.
- **Idea of proof:** Try to show A is independent of itself, i.e., that $P(A) = P(A \cap A) = P(A)P(A)$. Start with measure theoretic fact that we can approximate A by a set A_n in σ-algebra generated by $X_1, \dots X_n$, so that symmetric difference of A and A_n has very small probability. Note that A_n is independent of event A'_n that A_n holds when $X_1, \dots, X_n$ and $X_{n_1}, \dots, X_{2n}$ are swapped. Symmetric difference between A and A'_n is also small, so A is independent of itself up to this small error. Then make error arbitrarily small.

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- **Idea of proof:** Hewitt-Savage implies the $\limsup S_n$ and $\liminf S_n$ are almost sure constants in $[-\infty, \infty]$. Note that if X_1 is not a.s. constant, then both values would depend on X_1 if they were not in $\pm\infty$

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Stopping time definition

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- In finance applications, T might be the time one sells a stock. Then this states that the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

Stopping time examples

- Let $A_1, \ldots$ be i.i.d. random variables equal to -1 with probability .5 and 1 with probability .5 and let $X_0 = 0$ and $X_n = \sum_{i=1}^{n} A_i$ for $n \geq 0$.

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- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
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- Answer: first four, not last two.

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- **Wald's equation:** Let X_i be i.i.d. with $E|X_i| < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = EX_1 EN$.

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- **Wald's second equation:** Let X_i be i.i.d. with $E|X_i| = 0$ and $EX_i^2 = \sigma^2 < \infty$. If N is a stopping time with $EN < \infty$ then $ES_N = \sigma^2 EN$.

Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step S_j independently changes by ± 1 according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?

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- What is $\mathbb{E}N$?

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Reflection principle

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- Try counting walks that *do* cross by giving bijection to walks from $(0, -x)$ to (n, y).

Ballot Theorem

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- Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.

Arcsin theorem

- Theorem for last hitting time.

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- Theorem for amount of positive positive time.