18.177 Problem Set 1

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In these exercises, $G = (V, E)$ is a finite connected graph with positive edge weights $(c(e))_{e \in E}$, and $P = (p_{xy})$ is the transition matrix $p_{xy} = c(x, y)/C_x$, where $C_x = \sum_z c(x, z)$, and we define $c(x, y) = 0$ if $(x, y) \notin E$.

1. Show that P has a unique *right* eigenvector with eigenvalue 1, and deduce that the stationary distribution is unique, i.e., there is a unique probability distribution π on V satisfying $\pi(x) = \sum_{y \in V} \pi(y) p_{yx}$ for all $x \in V$.

In the remaining exercises, assume $c(e) = 1$ for all $e \in E$.

2. (Lyons & Peres, Ex. 2.53) Given $Z \subset V$ and a function $f_0 : Z \to \mathbb{R}$, let $\mathcal{F}$ be the set of all functions $f : V \to \mathbb{R}$ satisfying $f|_Z = f_0$. Show that the Dirichlet energy

$$(\nabla f, \nabla f) = \sum_{(x,y) \in E} (f(x) - f(y))^2$$

is uniquely minimized on $\mathcal{F}$ by the function h that is harmonic on Z^c. (*Hint*: Use the adjointness of div and ∇ to show that $(\nabla h, \nabla(f - h)) = 0$ for all $f \in \mathcal{F}$.)

3. (Lyons & Peres, Ex. 4.1) Fix $a \neq z \in V$, and let (Y_t) be the loop-erased random walk from a to z. Show that for any $y_0, y_1, \ldots, y_t, y \in V$ with $y_0 = a$ and $y_t \neq z$,

$$\mathbb{P}(Y_{t+1} = y \,|\, Y_0 = y_0, \ldots, Y_t = y_t) = \mathbb{P}_{y_t}\left(X_1 = y \,\middle|\, \tau_z < \tau^+_{\{y_0, \ldots, y_t\}}\right)$$

where $(X_t)_{t \geq 0}$ is simple random walk on G, and

$$\tau_z = \min\{t \geq 0 | X_t = z\}$$

$$\tau^+_{\{y_0, \ldots, y_t\}} = \min\{t \geq 1 | X_t \in \{y_0, \ldots, y_t\}\}.$$

4. (Lyons & Peres, Ex. 4.22) Let L be length of the path from vertex 1 to vertex 2 in the uniform spanning tree on the complete graph K_n. Show that for $\ell = 1, \ldots, n-1$

$$\mathbb{P}(L = \ell) = \frac{\ell + 1}{n} \prod_{i=1}^{\ell - 1} \frac{n - i - 1}{n}.$$

5. Fix a nonempty set $Z \subset V$. A *spanning forest* F rooted at Z is a disjoint union of trees F_z rooted at vertices $z \in Z$, whose vertex sets partition V.

   Show that if $F = \bigcup F_z$ is a uniform spanning forest rooted at Z, then for any $x \in V$
   $$\mathbb{P}(x \in F_z) = \mathbb{P}_x(X_\tau = z)$$
   where $(X_t)_{t \geq 0}$ is simple random walk on G, and
   $$\tau = \min\{t \geq 0 | X_t \in Z\}$$
   is the first hitting time of Z.