# 18.177 PROBLEM SET TWO, DUE MARCH 18

1. Let $\Omega$ be the collection of subsets of edges of $\mathbb{Z}^2$ and let $p$ and $q$ be distinct numbers in $(0,1)$. Let $P_{p,q}$ be the probability measure on $\Omega$ that independently includes each edge with probability $p$ if it is a vertical edge and probability $q$ if it is a horizontal edge. This is a generalization of ordinary bond percolation on $\mathbb{Z}^2$ (which would assume $p = q$). Carefully review the proofs of the following (given in class and/or in Grimmett's *Percolation* for the $p = q$ case) and explain (with at least a few lines) why each of the following holds or fails to hold in this generalized setting.

1. FKG inequality
2. Russo's formula
3. Zhang's argument for non-coexistence of infinite cluster and infinite dual cluster.
4. Burton-Keane argument for uniqueness of infinite open cluster.
5. Continuity (in both $p$ and $q$) of the probability $\theta(p,q)$ that the cluster $C$ containing the origin is infinite.
6. Exponential decay in the law of the radius of $|C|$.

Extend Kesten's theorem by describing the critical curve in $[0,1] \times [0,1]$ that gives the boundary of $\{(p,q) : \theta(p,q) = 0\}$.

2. Consider ordinary $p$-Bernoulli bond percolation on $\mathbb{Z}^3$. When $p = p_c$, it is unknown whether there exists an infinite cluster almost surely. In other words, it is unknown whether $\theta(p_c) > 0$, although it is generally believed that $\theta(p_c) = 0$. Some attempts to prove that $\theta(p_c) = 0$ involve assuming that $\theta(p_c) > 0$ and attempting to derive a contradiction. Prove the following, under the assumption that $\theta(p_c) > 0$.

1. For fixed vectors $\alpha, \beta \in [0,1]^3$, the probability that $n\alpha$ (components rounded down to integer parts) is connected to $n\beta$ by an open path contained in the box $[0,n]^3$ has a limsup strictly less than $\theta(p_c)^2$, as $n \to \infty$.

2. If $1 < a < 2$ is fixed, then the probability that there is an open path from $(0, 0, n)$ to *some* point in $\mathbb{Z} \times \mathbb{Z} \times \{an\}$ that does not cross $\mathbb{Z} \times \mathbb{Z} \times \{0\}$ tends to $\theta(p_c)$.