18.177: Lecture 7

Critical percolation

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Outline

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Other techniques

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- **Possible project:** just take one scenario and show it leads to one of the two problems above.

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- If $p = p_c$, we can construct a robust cluster in a slab (using full boxes and seeds)... if we sprinkle in a few more edges.
- In both cases, we explore and find an object that looks like super-critical infinite cluster — except that each is replaced by a long path of edges. As long as there is an upper bound on the lengths of these paths, the construction is robust.

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- How about a large origin-centered slab?

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- Represent paths as dots and pull them “taut”.

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- Can we arrange so that it maps boundary to boundary in facet preserving way (e.g., taking set where $p_2 = p_4 = p_5 = 0$ to corresponding set)?
- If so, must there be a point mapping to center of the simplex?

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- Generally, how many clusters are there with a given β ratio and $|\Lambda|$ value. Should grow exponentially in $|\Lambda|$ but how fast?
- See Hammond paper.