18.177: Lecture 5

Critical percolation

Scott Sheffield

MIT

Outline

Recollections

What else can we try?

Renormalization stories: percolation in slabs and half spaces

Outline

Recall

- **Basic tools:** Zero-ones, ergodic theorem, FKG inequality, BK inequality, Russo's formula.

Recall

- **Basic tools:** Zero-ones, ergodic theorem, FKG inequality, BK inequality, Russo's formula.
- **FKG "square root" trick:** If k increasing events are equally likely, and it is *very* likely that at least one occurs, then it is very likely that they all occur.

Recall

- **Basic tools:** Zero-ones, ergodic theorem, FKG inequality, BK inequality, Russo's formula.
- **FKG "square root" trick:** If k increasing events are equally likely, and it is *very* likely that at least one occurs, then it is very likely that they all occur.
- **Logarithmic Russo's formula:** $\frac{\partial}{\partial p} \log P_p(A) = \frac{P_p'(A)}{P_p(A)} = \frac{1}{p} E_p(N(A)|A)$.

Recall

- **Basic tools:** Zero-ones, ergodic theorem, FKG inequality, BK inequality, Russo's formula.
- **FKG "square root" trick:** If k increasing events are equally likely, and it is *very* likely that at least one occurs, then it is very likely that they all occur.
- **Logarithmic Russo's formula:**
$\frac{\partial}{\partial p} \log P_p(A) = \frac{P_p'(A)}{P_p(A)} = \frac{1}{p} E_p(N(A)|A)$.
- What games can you play using just these fundamental tools?

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^4 0$, say) that infinite cluster hits S_K.

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.
- So set of edges reachable by open paths in S_N starting from edges in S_K has one big piece with high probability.

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.
- So set of edges reachable by open paths in S_N starting from edges in S_K has one big piece with high probability.
- Now space out disjoint copies of S_K periodically throughout $\mathbb{Z}^3$ (at distance M from each other, day).

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.
- So set of edges reachable by open paths in S_N starting from edges in S_K has one big piece with high probability.
- Now space out disjoint copies of S_K periodically throughout $\mathbb{Z}^3$ (at distance M from each other, day).
- Is each big piece highly likely to be joined to next piece over within the box of radius $5M$?

Recall: one cluster in one big box?

- Suppose there exists an infinite cluster when $p = p_c$.
- Consider S_K where K is so large that it is likely (probability at least $p_E = 1 - 10^40$, say) that infinite cluster hits S_K.
- Now consider an N large enough so that it is extremely likely (prob at least p_E) that all vertices in S_K are either in finite clusters of diameter at most N or can be joined together by a path within a box of size S_N.
- So set of edges reachable by open paths in S_N starting from edges in S_K has one big piece with high probability.
- Now space out disjoint copies of S_K periodically throughout $\mathbb{Z}^3$ (at distance M from each other, day).
- Is each big piece highly likely to be joined to next piece over within the box of radius $5M$?
- Peierls argument: if so, could decrease p a bit and still have infinite cluster.

Outline

Outline

Recollections

What else can we try?

Renormalization stories: percolation in slabs and half spaces

Boundary hitting set

- FKG and square root trick: highly likely to have a path from nugget at zero to each of the $2d$ faces around it.

Boundary hitting set

- FKG and square root trick: highly likely to have a path from nugget at zero to each of the $2d$ faces around it.
- Question: can we find a way to come up with more than $2d$ faces that are all equally likely?

Boundary hitting set

- FKG and square root trick: highly likely to have a path from nugget at zero to each of the $2d$ faces around it.
- Question: can we find a way to come up with more than $2d$ faces that are all equally likely?
- Question: can we refine this and come up with some shape S (something sphere-like) such that the set of points on S we can reach from origin nugget in $\frac{1}{n}\mathbb{Z}^d$ becomes dense with high probability?

Boundary hitting set

- FKG and square root trick: highly likely to have a path from nugget at zero to each of the $2d$ faces around it.
- Question: can we find a way to come up with more than $2d$ faces that are all equally likely?
- Question: can we refine this and come up with some shape S (something sphere-like) such that the set of points on S we can reach from origin nugget in $\frac{1}{n}\mathbb{Z}^d$ becomes dense with high probability?
- Are there multiple large clusters within a large box that are in some sense space-filling?

Boundary hitting set

- FKG and square root trick: highly likely to have a path from nugget at zero to each of the $2d$ faces around it.
- Question: can we find a way to come up with more than $2d$ faces that are all equally likely?
- Question: can we refine this and come up with some shape S (something sphere-like) such that the set of points on S we can reach from origin nugget in $\frac{1}{n}\mathbb{Z}^d$ becomes dense with high probability?
- Are there multiple large clusters within a large box that are in some sense space-filling?
- Are there long paths of any given homotopy class (if we consider box minus some paths) or passing through given sequence of smaller boxes?

Slab crossing

- Suppose $p = p_c$ and consider the density of crossings of a slab of width k.

Slab crossing

- Suppose $p = p_c$ and consider the density of crossings of a slab of width k.
- How does this change as k increases?

Slab crossing

- Suppose $p = p_c$ and consider the density of crossings of a slab of width k.
- How does this change as k increases?
- Are certain clusters really very likely to join with at least one other cluster when we double or triple k?

Outline

Outline

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of {Grimmett, Marstrand, Barsky, Newman} have dealt with percolation on slabs and half spaces.

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of {Grimmett, Marstrand, Barsky, Newman} have dealt with percolation on slabs and half spaces.
- Write $p_c^{\text{slab}} = \lim_{k\to\infty} p_c(T_k)$ where T_k is the slab of width k.

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of $\{$Grimmett, Marstrand, Barsky, Newman$\}$ have dealt with percolation on slabs and half spaces.
- Write $p_c^{\text{slab}} = \lim_{k\to\infty} p_c(T_k)$ where T_k is the slab of width k.
- Theorem: $p_c^{\text{slab}} = p_c$.

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of {Grimmett, Marstrand, Barsky, Newman} have dealt with percolation on slabs and half spaces.
- Write $p_c^{\text{slab}} = \lim_{k\to\infty} p_c(T_k)$ where T_k is the slab of width k.
- Theorem: $p_c^{\text{slab}} = p_c$.
- Generally, if F is any infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$, then each $\eta > 0$ there exists an integer k such that $p_c(2kF + B(k)) \leq p_c + \eta$.

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of {Grimmett, Marstrand, Barsky, Newman} have dealt with percolation on slabs and half spaces.
- Write $p_c^{\text{slab}} = \lim_{k\to\infty} p_c(T_k)$ where T_k is the slab of width k.
- Theorem: $p_c^{\text{slab}} = p_c$.
- Generally, if F is any infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$, then each $\eta > 0$ there exists an integer k such that $p_c(2kF + B(k)) \leq p_c + \eta$.
- Idea: just start building things out of blocks and "seeds".

Percolation in slabs (Grimmett Chapter 7)

- Important works by subsets of $\{$Grimmett, Marstrand, Barsky, Newman$\}$ have dealt with percolation on slabs and half spaces.
- Write $p_c^{\text{slab}} = \lim_{k\to\infty} p_c(T_k)$ where T_k is the slab of width k.
- Theorem: $p_c^{\text{slab}} = p_c$.
- Generally, if F is any infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$, then each $\eta > 0$ there exists an integer k such that $p_c(2kF + B(k)) \leq p_c + \eta$.
- Idea: just start building things out of blocks and "seeds".
- Need sprinkling to compensate for "negative information".

Percolation in half spaces (Grimmett Chapter 7)

- Theorem: When $p = p_c$ there is no percolation in the half plane.

Percolation in half spaces (Grimmett Chapter 7)

- Theorem: When $p = p_c$ there is no percolation in the half plane.
- Idea: again, start building things out of blocks again, but now we don't need sprinkling.