18.177: Lecture 3

Critical percolation

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Outline

Recollections

Exponential decay

Intuition when d is very large

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- **Consequence of FKG:** Can't have both infinite cluster/dual-cluster when $d = 2$. Thus $\theta(1/2) = 0$, $p_c \geq 1/2$.

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- You say, "There's at least a tiny positive chance that there's a squirrel somewhere."
- I say, "Any sufficiently large box has probability at least .99999 of being infested by positive density of squirrels."

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Exponential decay in sub-critical regime

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- This claim now implies Kesten's theorem, that $p_c = 1/2$.
- Proof requires some new tools.

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- Thus $\frac{\partial}{\partial p} P_p(A)$ is $p^{-1}E_p(N(A); A)$.

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- $g_\alpha(n) = g_\beta(n) \exp\left(-\int_\alpha^\beta \frac{1}{p} E_p(N(A_n)|A_n) dp\right)$
- If we can can show $E_p(N(A_n)|A_n)$ grows roughly linearly in n when $p < p_c$ (the bound should hold uniformly for an interval of p values), then this will imply that when $p < p_c$ there is a $\psi(p) > 0$ such that $P_p(A_n) < e^{-n\psi(p)}$.

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- Write $M = \max\{k : A_k \text{ occurs}\}$. Idea: try to show that number $N(A_n)$ (conditioned on A_n) is at least as large as number of renewals of renewal process whose elements have approximately same distribution as M. We'd like the individual sausages to be smaller than copies of M.

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- It's kind of annoying that we don't even know a priori that M has finite expectation. We'll have to find some sort of bootstrapping trick for getting around this eventually.

A nice lemma involving M

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- Lemma: fix $k > 0$, integers $r_1, r_2, \ldots r_k$ such that $\sum_{i=1}^k r_i \le n - k$. Then

$$P_p(\rho_k \le r_k, \rho_i = r_i \text{ for } 1 \le i < k | A_n) \ge$$

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- We have to have at least two disjoint paths up to starting point of first pivotal edge. BK inequality implies $P_p(\{\rho_1 > r_2\} \cap A_n) \le P_p(A_{r_1+1}) P_p(A_n)$.

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- Extend to the general case.

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$$E_p(N(A_n)|A_n) \geq \sum_{k=1}^{\infty} P(M_1 + \ldots + M_k \leq n)$$

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where $K = \min\{k : M_1 + \ldots + M_k > n\}$.
- $E(K) > \frac{n}{E(M_1)} = \frac{n}{1 + E(min\{M_1, n\})} = \frac{n}{\sum_{i=0}^n g_p(i)}$.

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- Plugging this into earlier formula lets us show that $\sum_{n=1}^{\infty} g_\alpha(n) < \infty$ for $\alpha < p_c$, and complete the exponential decay proof.

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- Expect to have lots of large tree like clusters intersecting the n^d box.
- Heuristically, tree with k vertices should have a longest path of length $\sqrt{k}$. Is distance of tip from origin about $k^{1/4}$?