18.177: Lecture 2

Critical percolation

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Outline

Review of miracles from last time with new details

FKG inequality and the case $d = 2$

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- Big question is whether $\theta(p_c) > 0$ when $d = 3$. (Answer is known only for $d = 2$ and $d \geq 19$.)

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- Let $E_n[A]$ be the conditional expectation of A given the values of ω on edges of radius n ball S_n centered at zero.
- For any A, $\lim_{n\to\infty} E_n[A] = 1_A$ almost surely.

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- **Ergodic theorem:** if $F : \Omega \to \mathbb{R}$ has finite expectation, then average of F, over the translations of ω by elements of S_n, P-a.s. tends to this expectation as $n \to \infty$.

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- If $\theta(p) = 0$ have a.s. zero infinite clusters. If $\theta(p) = 1$ have a.s. one infinite cluster.
- **By ergodic theorem:** asymptotic density of infinite cluster is a.s. $\theta(p)$.

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- **Continuity:** Can now show continuity of θ on $(p_c, 1]$.

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- **Proof:** Simple induction applies if random variables depend on finitely many edges.
- **Proof:** More generally, let X_n and Y_n be conditional expectations given first n edges in enumeration of edges. Then $X_n \to X$ and $Y_n \to Y$ a.s. by martingale convergence (and in $L^2(P_p)$). Take limits.

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- BK inequality says that the probability that A and B occur disjointly is *most* $P(A)P(B)$.

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- In particular, symmetry implies that a.s. we have no infinite cluster when $p = 1/2$.
- But this doesn't quite prove $p = p_c$. Could there be a range of p values for which there is neither an infinite cluster nor an infinite dual cluster?