18.177: Lecture 1

Critical percolation

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Outline

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- **Percolation probability:** Write $\theta = \theta(p) = P_p(|C| = \infty)$.

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- $\mathcal{F}$ is the σ-algebra generated by finite dimensional cylinders of Ω and P_p is the product measure on $(\Omega, \mathcal{F})$.

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- p_c **upper bound:** Peierls argument shows $p_c(2) < 1 - 1/\lambda(2)$ (hence $p_c(d) < 1 - 1/\lambda(2)$ when $d \geq 2$).

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- **Burton-Keane argument (next slide):** The number of infinite clusters is a.s. not ∞, for any p.

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- Expected number of trifurcations in box $[1, n]^d$ grows like n^d time a constant.
- But combinatorial argument shows that the possible number of trifurcations in the box goes like surface area of the box.
- Conclude that we almost surely don't have infinitely many clusters.

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- Unknown when $3 \leq d \leq 18$, but people seem convinced that $\theta(p_c) = 0$ for all d. Why?

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FKG inequality

More to come

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- **Proof:** Simple induction applies if random variables depend on finitely many edges.
- **Proof:** More generally, let X_n and Y_n be conditional expectations given first n edges in enumeration of edges. Then $X_n \to X$ and $Y_n \to Y$ a.s. by martingale convergence (and in $L^2(P_p)$). Take limits.

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- If we did, then by FKG and symmetry, we would have a high probability of having a path going from each of top and bottom of a box to infinity, and a dual path going from each of left and right side of box to infinity. This implies there has to be either more than one infinite cluster or more than one infinite dual cluster.

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- We will use this to prove Kesten's theorem: that $p_c = 1/2$ when $d = 2$.
- Barsky, Grimmett, Newman: *if* there is an infinite cluster when $p = p_c$ for any $d \geq 3$ then it has to be fairly strange. There is no infinite cluster on any half space when $p = p_c$, so any path to infinity has to oscillate back and forth a lot.