MATH 177 PROBLEM SET 2

A. Answer the following DGFF questions:

1. Let G be the graph with vertices $\{0, 1, 2, \dots, n\}$, where i and j are adjacent when $|i - j| = 1$. Designate 0 as the boundary vertex, and let h an instance of the discrete Gaussian free field on G with zero boundary conditions (i.e., $h(0) = 0$ with probability one). Prove that the increments $h(j + 1) - h(j)$ are independent Gaussian random variables and write an explicit formula for the probability density function (a real-valued function on $\mathbb{R}^n$) of the n-tuple $(h(1), h(2), \dots, h(n))$.

2. Suppose we instead took 0 and n both to be boundary vertices and set $h(0) = h(n) = 0$. Compute the covariance of $h(i)$ and $h(j)$ in this case.

3. Now let G be the graph whose vertices are all non-negative integers, where i and j are adjacent when $|i - j| = 1$. Designate 0 as the boundary vertex, and let h be an instance of the discrete Gaussian free field on G with zero boundary conditions. This is a standard Gaussian on an infinite dimensional vector space. Since the space of functions is infinite dimensional, we can write $h = \sum \alpha_i f_i$ where f_i are an orthonormal basis for this vector space under the Dirichlet inner product and α_i are mean zero, variance one normal random variables. Prove that this sum converges almost surely point-wise, no matter what orthonormal basis is used. Prove that the increments $h(j + 1) - h(j)$ are independent Gaussian random variables. [Extra credit: use a similar argument to show that the continuum GFF on the real open interval $[0, \infty)$, with 0 as the sole boundary vertex, is just Brownian motion. You can start by considering the subspace of Hilbert space comprised of functions that are affine on each interval $[j, j + 1]$ for integer j.]

4. Let h be an instance of the DGFF on a general graph G, and suppose that v is a non-boundary vertex of G with n neighbors. Let X be $h(v)$ minus the average value of h on the set of neighbors of v. Show that the mean and variance of X depend on n only and compute these quantities as functions of n.

B. Let h be an instance of the Gaussian free field on the unit disc $\mathbb{D}$ with zero boundary conditions. Let $h_\epsilon(z)$ denote the mean value of h on a circle of radius ϵ.

1. Use integration by parts to prove the identity $(f,g)_\nabla = (f,-\Delta g) = (-\Delta f, g)$ when f and g are smooth and compactly supported on $\mathbb{D}$.

2. Express the random variable $h_\epsilon(z)$ as a Dirichlet inner product $(h,\rho)_\nabla$ for some function ρ on $\mathbb{D}$ (depending on z and ϵ) whose Dirichlet energy is finite.

3. Compute the variance of $h_\epsilon(0)$ as a function of ϵ. Can you generalize from 0 to other values of z?

4. Show that $h_{e^{-t}}(0)$, viewed as a function of t, is a Brownian motion.

5. Compute the covariance of $h_{\epsilon_1}(z_1)$ and $h_{\epsilon_2}(z_2)$ under the assumption that the balls $B_{\epsilon_1}(z_1)$ and $B_{\epsilon_2}(z_2)$ are disjoint and contained in $\mathbb{D}$. Write $A_\epsilon(z) = e^{h_\epsilon(z)}$, and compute $\mathbb{E}A_{\epsilon_1}(z_1)A_{\epsilon_2}(z_2)$ under the same assumption.

C. Suppose that X is a random variable on $\mathbb{R}^2$ and that for each deterministic $Y \in \mathbb{R}^2$ the random variable (X,Y) (where $(\cdot,\cdot)$ is the usual inner product on $\mathbb{R}^2$) is a normal random variable with mean zero and variance (Y,Y).

1. Prove that X is a standard Gaussian on $\mathbb{R}^2$. (Hint: what can you say about the characteristic function $\phi(Y) := \mathbb{E}e^{i(X,Y)}$? What about the Fourier transform of the density function of X?)

2. Does this argument generalize from $\mathbb{R}^2$ to $\mathbb{R}^n$? What about infinite dimensions?

D. Read and thoroughly understand "Gaussian free fields for mathematicians" and the first two (plus what you can of the first four) sections of "Liouville Quantum Gravity and KPZ". Read Sections 1.1 to 1.4 and Sections 2 through 4 of "Conformal weldings of random surfaces: SLE and the quantum gravity zipper". Read "Quantum gravity and inventory accumulation". (All papers available at arXiv.org.)

E. Let us understand the free boundary GFF:

1. Show that set of real valued functions on a planar domain D, modulo additive constants, is a Hilbert space under the Dirichlet inner product. The free boundary GFF can be defined as $h = \sum \alpha_i f_i$ where the α_i are i.i.d. normal random variables and the f_i are an orthonormal basis for this Hilbert space.

2. Show that (h, ρ) is almost surely well defined for any smooth compactly supported function ρ with mean zero.

3. As in the fixed boundary case, show that there exists a function $G(x, y)$ such that

$$\mathrm{Cov}((h, \rho_1), (h, \rho_2)) = \int_D \rho_1(x) G(x, y) \rho_2(y) dx dy.$$

   Call this the free boundary Green's function.

4. Compute explicitly both the free boundary and zero boundary Green's function for the half plane $\mathbb{H}$.

5. How does Green's function transform under conformal maps? If ϕ is a conformal map from D to D', how are $G^D(x, y)$ and $G^{D'}(\phi(x), \phi(y))$ related? Is the answer the same for free and fixed boundary?

F. Let h be an instance of the free boundary GFF on $\mathbb{H}$. Compute explicitly the variance of $h_2(2) - h_1(2)$, where in this context $h_\epsilon(z)$ denotes the mean value of h on $\partial B_\epsilon(z) \cap \mathbb{H}$. Compute explicitly the variance of $h_1(1) - h_1(3)$.