$\iota : \partial D \hookrightarrow X$ inclusion map, $\mu \in \Omega_c^{n-1}(X)$.

Thm. (Stoke's' Theorem) $\forall \mu \in \Omega_c^{n-1}(X)$, we have

$$\int_{\partial D} \iota^* \mu = \int_D d\mu.$$

proof. We can cover X by parametrizable open subsets $\{U_\alpha\}_{\alpha \in I}$.

w.l.o.g., we can assume that the diffeomorphism

$$\phi_\alpha : U_\alpha \to X, \text{ if } \phi_\alpha(U_\alpha) \cap \partial D \neq \phi,$$

satisfies $\begin{cases} \phi_\alpha(U_\alpha \cap \mathbb{H}_{\{x_n \le 0\}}) \subseteq D \\ \text{with } \phi_\alpha(U_\alpha \cap (\mathbb{R}^{n-1} \times \{0\})) \subseteq \partial D \\ \phi_\alpha(U_\alpha \cap \mathbb{H}_{\{x_n \ge 0\}}) \subseteq X \setminus D. \end{cases}$

Then $\cancel{\int_D d\mu}$

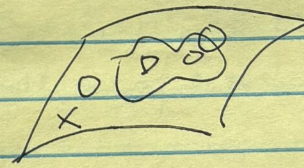
$$\cancel{= \sum_{i=1}^{\infty} \int_D \rho_i \, d\mu} \quad \left(\text{because } \sum_{i=1}^{\infty} \rho_i = 1\right)$$

$$\int_D d\left(\sum_{i=1}^{\infty} \rho_i \mu\right)$$

$$= \sum_{i=1}^{\infty} \int_{(U_i \cap D)} d(\rho_i \mu).$$

If $U_i \subseteq \mathrm{int}(D)$ or $U_i \subseteq X \setminus D$,

then $\int_{(U_i \cap D)} d(\rho_i \mu) = 0$.



If $U_i \cap \partial D \neq \phi$, then

$$\int_{(U_i \cap D)} d(\rho_i \mu) = \int_{\mathbb{H}_{\{x_n \le 0\}}} d(\phi_i^*(\rho_i \mu))$$

$$= \int_{\mathbb{R}^{n-1}} \phi_i^* \rho_i \mu. \quad \text{(by what we learned before).}$$

Then $\{U_i \cap \partial D\}$ w/ $U_i \cap \partial D \neq \phi$ forms an open cover of ∂D,

so $\sum_{i=1}^{\infty} \int_{\mathbb{R}^{n-1}} \phi_i^* \rho_i \mu = \sum_i \int_{\partial D} \rho_i \mu = \int_{\partial D} \mu$. $\square$

Cor. $\int_D L_v\omega = \int_{\partial D} \iota_{\partial D}^*(\iota_v\omega)$.

proof. $\int_D L_v\omega = \int_D (\iota_v d\omega + d\iota_v\omega)$

$$= \int_D d\iota_v\omega = \int_{\partial D} \iota_{\partial D}^* \iota_v\omega . \quad \square$$

§4.7 Degree theory on manifolds.

Thm. Let X be a connected n-dim mfld, oriented.
Then given any $\omega \in \Omega_c^n(X)$ s.t. $\int_X \omega = 0$,
we can find $\mu \in \Omega_c^{n-1}(X)$ such that $\omega = d\mu$.

proof. Step 1. We show that for $U_0, U_1 \subseteq X$ parametrizable open subsets,
given $\omega_0 \in \Omega_c^n(U_0)$ s.t. $\int_{U_0}\omega_0 = 1$,
if $\omega_1 \in \Omega_c^n(U_1)$ and $\int_{U_1}\omega_1 = c$, then $c\omega_0 - \omega_1$ is exact.

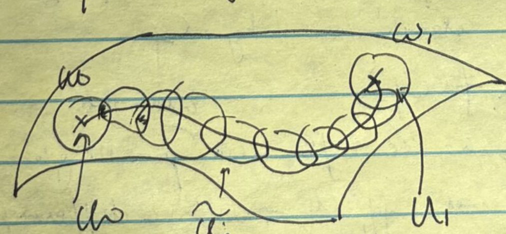
When $U_0 = U_1$, this is the Poincaré Lemma in $\mathbb{R}^n$.

In general. Claim. Given U_0, U_1, $\exists$ finitely many parametrizable open subsets $U_0, \tilde{U}_1, \dots, \tilde{U}_N, U_1$ such that

$U_0 \cap \tilde{U}_1 \neq \emptyset$, $\tilde{U}_i \cap \tilde{U}_{i+1} \neq \emptyset$ $(1 \le i \le N-1)$, $\tilde{U}_N \cap U_1 \neq \emptyset$.

↳ This follows from choosing a finite cover of a path joining U_0 and U_1.

Then we can repeat the argument in $\mathbb{R}^n$.



Step 2. Choose parametrizable open subsets $U_1, \dots, U_N$ covering $\mathrm{supp}(\omega)$.
Find a subordinate partition of unity $\rho_i \in C_0^\infty(X)$.
Write $c_i = \int_X \rho_i\omega$, then we know that $\rho_i\omega = c_i\omega_0 + d\mu_i$.

$\Longrightarrow \quad \omega = (\sum \rho_i)\omega$
$= \sum \rho_i \omega$
$= \sum c_i \omega_0 + \sum d\mu_i$
$= (\sum c_i)\omega_0 + \sum d\mu_i = d(\sum \mu_i) \to \mu$. $\square$

Thm. Let X, Y be oriented n-mflds w/. Y connected.
Suppose $f: X \to Y$ is a proper C^∞-map.
Then there exists $\deg(f) \in \mathbb{Z}$ s.t. $\forall \omega \in \Omega_c^n(Y)$,
we have $\int_X f^*\omega = \deg(f) \int_Y \omega$.

proof. We choose $\omega_0 \in \Omega_c^n(Y)$ s.t. $\int_Y \omega_0 = 1$.
Then define $\deg(f) = \int_X f^*\omega_0$.

Well-definedness: if $\int_Y \omega_0' = 1$, then $\exists \mu \in \Omega_c^{n-1}(Y)$ s.t. $\omega_0 - \omega_0' = d\mu$.
So $\int_X f^*\omega_0' = \int_X f^*\omega_0 + \int_X df^*\mu = \int_X f^*\omega_0$.

Property: We know that $\omega - (\int_Y \omega)\omega_0 = d\mu'$,
so $\int_X f^*\omega = (\int_Y \omega)(\int_X f^*\omega_0) + \int_X df^*\mu' = \deg(f)\int_Y \omega$. $\square$

Prop^n. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ w/. f, g proper, we have
$\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

proof. $\deg(g \circ f) = \int_X (g \circ f)^*\omega_0$
$= \int_X f^*(g^*\omega_0) = \deg(f)\int_Y g^*\omega_0 = \deg(f)\deg(g)$. $\square$

Def^n. A point $p \in X$ is a critical point of f if $df_p: T_pX \to T_{f(p)}Y$ is not surjective.
denote the set by C_f

Regular value: $Y \setminus f(C_f)$.

Thm. Give $q \in Y \setminus f(C_f)$, we can find an connected open neighborhood V of q s.t.
$f^{-1}(V) = \coprod U_p$, and $f|_{U_p}: U_p \to V$ is a diffeomorphism.

Choosing $\omega \in \Omega_c^n(V)$ s.t. $\int_V \omega = 1$,
then $\int_{U_p} f^*\omega = 1$ if df_p is orientation-preserving
$\int_{U_p} f^*\omega = -1$ if df_p is orientation-reversing.

Thm. $f: X^n \to Y^n$ smooth proper between oriented mfds. If $q \in Y$ is a regular value, then
$\deg(f) = \sum_{p \in f^{-1}(q)} \operatorname{sign}(df_p)$. $\square$

Claim. The proof of Sard theorem work for mflds.
↳ Just work locally over parametrizable open subsets.

Def^n. Let X, Y be mflds and $f_0, f_1 : X \to Y$ be C^∞-maps.
A C^∞-map $F : X \times [0,1] \to Y$ is a homotopy between f_0 and f_1 if
$\forall x \in X$, we have $F(x,0) = f_0(x)$, $F(1,x) = f_1(x)$.
F is called a proper homotopy if F is a proper map.

Prop^n. Two properly homotopic proper maps have the same degree.
proof. The function $t \mapsto \deg(F(-,t)) \in \mathbb{Z}$
is continuous because R.H.S. $= \int_X F(-,t)^* \omega_0$.

As $\mathbb{Z}$ is discrete, the map is constant. □