§ 4.4 Orientation

<u>observation</u>. Let $U \subseteq X$ be an open subset of a mfld. Then the space $\begin{array}{c} U \times \mathbb{R}^n \\ \downarrow \pi \\ U \end{array}$ with the projection

has the property that: ① $\exists$ an addition operation covering $U \xrightarrow{Id} U$,

$(p, a) + (p, b) = (p, a+b)$.

② $\exists$ a scaler multiplication covering $U \xrightarrow{Id} U$

$\lambda \cdot (p, a) = (p, \lambda a)$

③ together w/. the point $(p, 0)$, each $\{p\} \times \mathbb{R}^n$ is a vector space.

We call such $\begin{array}{c} U \times \mathbb{R}^n \\ \downarrow \pi \\ U \end{array}$ a trivial vector bundle.

<u>Def^n</u>. Let X be a mfld. A vector bundle $\begin{array}{c} E \\ \downarrow \pi \\ X \end{array}$ of rank n is a mfld & a smooth map π such that

① $\exists$ open covering $X = \bigcup_\alpha U_\alpha$ s.t. $\exists$ diffeomorphism s.t.

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^n \arrow[r, "\varphi_\alpha"] \arrow[d, "\pi_{U_\alpha}"'] & E|_{U_\alpha} := \pi^{-1}(U_\alpha) \arrow[d, "\pi"] \\ U_\alpha \arrow[r, "Id"] & U_\alpha \end{tikzcd}$$

commutes.

② $\forall \alpha, \beta$ s.t. $U_\alpha \cap U_\beta \neq \phi$, the map $(p, x) \mapsto (p, g_{\alpha\beta}(p)x)$

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times \mathbb{R}^n \arrow[r, "\varphi_\alpha|_{U_\alpha \cap U_\beta}"] \arrow[d, "\pi_{U_\alpha}"'] \arrow[rr, bend left] & E|_{U_\alpha \cap U_\beta} \arrow[d] & (U_\alpha \cap U_\beta) \times \mathbb{R}^n \arrow[l, "\varphi_\beta|_{U_\alpha \cap U_\beta}"'] \arrow[d, "\pi_{U_\beta}"] \\ U_\alpha \cap U_\beta \arrow[r, "Id"] & U_\alpha \cap U_\beta & U_\alpha \cap U_\beta \arrow[l, "Id"'] \end{tikzcd}$$

diffeomorphism

where $g_{\alpha\beta}(p)$ is the ~~map~~ of $\mathbb{R}^n$ $\varphi_\beta^{-1} \circ \varphi_\alpha \big|_{\{p \in U_\alpha \cap U_\beta\} \times \mathbb{R}^n}$ is a linear isomorphism.

<u>Rmk</u>. Then we can define fiberwise + & scaler multiplication.

Example. $\begin{array}{c} TX \\ \downarrow \\ X \end{array}$ tangent bundle.

Let $\phi_0 : U_0 \to U \subseteq X$ be a parametrizable open subset.
Then $TU_0 = U_0 \times \mathbb{R}^n$ as $U_0 \subseteq \mathbb{R}^n$ open.
For U_0, U_0' w/. parametrization ϕ_0, ϕ_0', if $U \cap U' \neq \phi$,
then look at

$$\begin{tikzcd}
U \times \mathbb{R}^n \arrow[r, "\phi_0^{-1} \times d\phi_0"] \arrow[d] \arrow[rrr, bend right=60, "{(Id,\ (d\phi_0')^{-1} \circ (d\phi_0))}"'] & U_0 \times \mathbb{R}^n \arrow[r, "\phi_0 \times d\phi_0"] \arrow[d] & TX|_U \arrow[d] & U_0' \times \mathbb{R}^n |_{U' \cap U} \arrow[l, "Id \times d\phi_0'"'] \arrow[d] \\
U \arrow[r, "\phi_0^{-1}"] & U_0 \arrow[r, "\phi_0"] & U & U' \arrow[l, "Id"']
\end{tikzcd}$$

$\Rightarrow$ vector bundle.

Example. $\begin{array}{c} \Lambda^k T^* X \\ \downarrow \\ X \end{array}$. In particular: $\begin{array}{c} \Lambda^n T^* X \\ \downarrow \\ X \end{array}$ for $\dim X = n$. has rank 1.

Def^n. X is called orientable if $\Lambda^n T^* X$ admits a nowhere zero $\boxed{\text{section}}$.

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "s"'] \end{tikzcd} \text{ s.t. } \pi \circ s = id_X.$$

Def^n. An orientation is ~~a choice~~ an assignment of "positive basis".
$\Downarrow$ saying which is the positive component.

Def^n. For $\dim X = n$, if $\omega \in \Omega^n(X)$ and $\omega_p \in (\Lambda^n T_p^* X)_+ \ \forall p \in X$,
we call ω a volume form.

Rmk. For ω_1, ω_2 volume forms, they differ by multiplying a positive function.

Example. For $U \subseteq \mathbb{R}^n$, its standard orientation is defined by

$$dx_1 \wedge \cdots \wedge dx_n.$$

Example. (Induced orientations).

Let $f:\mathbb{R}^N\to\mathbb{R}^k$ be a C^∞ map, suppose 0 is a regular value.

Then $X=f^{-1}(0)$ is a smooth mfld.

Its tangent space fits into

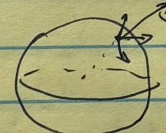
$$\cancel{0\to\mathbb{R}^k\to T_pX}\quad 0\to T_pX\hookrightarrow\mathbb{R}^N\xrightarrow{df}\mathbb{R}^k\to 0$$

as $\forall p\in X=f^{-1}(0)$, $T_pX=\ker(df_p)$.

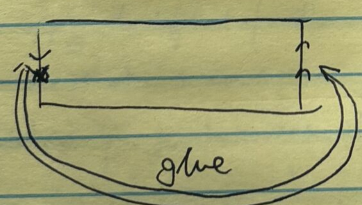
It admits an orientation s.t. the positive generator ω^+ satisfies "$\omega^+\wedge dy_1\wedge\cdots\wedge dy_k=dx_1\wedge\cdots\wedge dx_N$".

$\Longrightarrow$ For $S^n\subseteq\mathbb{R}^{n+1}$ defined by $f(x_1,\dots,x_{n+1})=x_1^2+\cdots+x_{n+1}^2-1=0$,

it has an induced orientation



Example. (Möbius strip)



This is not orientable because gluing the ∂ reverses the orientation.

Def^n. Let X,Y be oriented. $f:X\to Y$, diffeomorphism is called orientation-preserving iff $\forall p$, $df_p:T_pX\to T_{f(p)}Y$ is orientation preserving.

Prop^n. If $X\underset{\sim}{\xrightarrow{f}}Y\underset{\sim}{\xrightarrow{g}}Z$ & f,g are orientation-preserving, so is $g\circ f$.

Def^n. A parametrization $\phi_0:U_0\to U$ is called an oriented parametrization if it's orientation-preserving.

Fact. If ϕ_0,ϕ_1 are oriented parametrization, $\det(d(\phi_1^{-1}\circ\phi_0))>0$.

Def^n. An open subset $D\subseteq X$ is called a smooth domain iff

(1) ∂D is an $(n-1)$-dim submfld of X and
~~(2) $\partial D = \partial \overline{D}$.~~ (2) $\partial D = \partial \overline{D}$.

Examples. (a) $B\ \{x_1^2 + \cdots + x_n^2 < 1\} \subseteq \mathbb{R}^n$, ∂ is S^{n-1}
(b) $\{1 < x_1^2 + \cdots + x_n^2 < 2\} \subseteq \mathbb{R}^n$. annulus, $\partial = S^{n-1} \cup S^{n-1}$.
(c) $\mathbb{R}^n \setminus S^{n-1}$ is NOT: $\overline{\mathbb{R}^n \setminus S^{n-1}} = \mathbb{R}^n$.
(d) $\mathbb{H}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 < 0\} \Rightarrow \partial \mathbb{H}^n = \{(x_1, \cdots, x_n) \mid x_1 = 0\}$.

Thm. Let D be a smooth domain and suppose $p \in \partial D$.
Then $\exists$ open nbhd of $p \in U$ & $U_0 \subseteq \mathbb{R}^n$ open, $\psi: U_0 \xrightarrow{\sim} U$ diffeomorphisms
s.t. $\psi: U_0 \cap \mathbb{H}^n \rightarrow U \cap D$.

proof. This a local statement: w.l.o.g. we assume that $D \subseteq \mathbb{R}^n$.
As ∂D is an $(n-1)$-dim mfld, $\forall p \in \partial D$, $\exists p \in U$ open,
and a smooth function $\phi: U \rightarrow \mathbb{R}$ s.t. $\partial D \cap U = \phi^{-1}(0)$ & $d\phi \neq 0$.
w.l.o.g. we can assume $\frac{\partial \phi}{\partial x_1} \neq 0$
The apply the implicit function theorem to $(x_1, \cdots, x_n) \longmapsto (\phi, x_2, \cdots, x_n)$,
we see that we can assume $\phi = x_1$.
To obtain ψ, just reverse the sign $(x_1, \cdots, x_n) \mapsto (-x_1, \cdots, x_n)$
if necessary to get the correct half space. $\square$

Lemma. ψ can be chosen to be orientation-preserving.
proof. If not, just flip the sign $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, -x_n)$. $\square$

Def^n. For a domain D, the induced orientation on ∂D is defined by asking that
for ψ as above, $\psi|_{\partial \mathbb{H}^n} \rightarrow \partial D$ is orientation-preserving.

Prop^n. This is well-defined.
proof. It suffices to show that for $U_0, U_1 \subseteq \mathbb{R}^n$ open
s.t. $f: U_0 \xrightarrow{\sim} U_1$ diffeomorphism w/ $U_0 \cap \mathbb{H}^n \rightarrow U_1 \cap \mathbb{H}^n$,
the restriction of f to $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$ defines an orientation-preserving diffeomorphism
$U_0 \cap \mathbb{R}^{n-1} \rightarrow U_1 \cap \mathbb{R}^{n-1}$.

Write $f = (f_1(x_1, \dots, x_n), \dots f_n(x_1, \dots, x_n))$,

then $f_1(0, x_2, \dots, x_n) = 0$,

and if $x_1 < 0$, $f_1(x_1, \dots, x_n) < 0$.

This shows that $\frac{\partial f_1}{\partial x_1}(0, x_2, \dots, x_n) > 0$.

So $\det(df)(0, x_2, \dots, x_n) = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \\ & \boxed{df|_{\mathbb{R}^{n-1}}} \end{pmatrix} (0, x_2, \dots, x_n) > 0$,

therefore $\det(df|_{\mathbb{R}^{n-1}}) > 0 \implies$ orientation-preserving. $\square$

Cor. Given oriented mflds X_1, X_2 and smooth domains $D_1 \subset X_1$, $D_2 \subset X_2$, if $f : X_1 \to X_2$ is an orientation-preserving diffeomorphism taking D_1 to D_2, then

$f|_{\partial D_1} : \partial D_1 \to \partial D_2$ is an orientation-preserving diffeomorphism. $\square$

§ 4.5 Integration on manifolds

$X \Rightarrow$ n-dim smooth mfld, oriented.

$\omega \in \Omega^n_c(X)$. More generally, ~~$W \subseteq X$ open, $\omega \in \Omega^n_c(W)$~~.

Let $\phi_0 : \mathbb{R}^n \supseteq U_0 \xrightarrow{\cong} U \subseteq X$ an oriented parametrization.

~~then define s.t. supp~~

~~$\int_W \omega$~~

Assuming $\operatorname{supp}(\omega) \subseteq U$, define
$$\int_U \omega = \int_{U_0} \phi_0^* \omega.$$

Here: Using coordinates on $\mathbb{R}^n$, write $\phi_0^* \omega = f(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$,

then $\int_U \omega := \int_{U_0} f(x_1, \dots x_n)\, dx_1 \dots dx_n$.

Lemma. This is well-defined.

proof. Given a different parametrization preserving the orientation, we have

$$\begin{tikzcd} U_0 \arrow[r, "\phi_0"] \arrow[d, "\phi_1^{-1} \circ \phi_0"'] & U \\ U_1 \arrow[ur, "\phi_1"'] & \end{tikzcd}$$

, then $\int_{U_1} \phi_1^* \omega$

$= \int_{U_0} (\phi_1^{-1} \circ \phi_0)^* \phi_1^* \omega$

$= \int_{U_0} \phi_0^* \omega$ by change-of-variable formula. $\square$

If U is parametrizable, for $\omega_i \in \Omega_c^n(U)$, $i=1,2$, we have

$$\begin{cases} \int_U (\omega_1+\omega_2) = \int_U \omega_1 + \int_U \omega_2 \\ \int_U c\cdot\omega_i = c\int_U \omega_i . \end{cases}$$

Def^n. Let X be a mfd. Suppose $\{U_\alpha\}_{\alpha\in I}$ is an open covering of X.
$\{\rho_i\}_{\cancel{\alpha\in I}\; i\in\mathbb{Z}_{\geq 1}}$ is called a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in I}$ if.

(1) $\rho_i \in C_0^\infty(X)$ & $\rho_i \geq 0$ everywhere.

(2) $\forall\ C\subseteq X$ compact, $\exists N$ s.t. $\forall i > N$, $\mathrm{supp}(\rho_i)\cap C=\phi$.

(3) $\sum_{i=1}^{N} \rho_i = 1$.

(4) $\forall i\in\mathbb{Z}_{\geq1}$, $\exists \alpha\in I$ such that $\mathrm{supp}(\rho_i)\subseteq U_\alpha$.

Prop^n. Given any open cover, such a partition always exists. $\square$

Def^n. Given $\omega\in\Omega_c^n(X)$, the integration

$$\int_X \omega \text{ is defined by}$$

(1) Choose an open cover $\{U_\alpha\}_{\alpha\in I}$ of X by parametrizable U_α.

(2) Chose a partition of unity as above,

$$\int_X \omega := \sum_{i=1}^{\infty} \int \rho_i \omega \Rightarrow \text{a finite sum because } \mathrm{supp}(\omega) \text{ is compact.}$$

Lemma. This is well-defined.

proof. Let $\{U'_\beta\}_{\beta\in J}$ be another covering

$\{\rho'_j\}_{j\geq1}$ be another partition of unity.

Then $\int_X \omega = \sum_{i=1}^{\infty}\int \rho_i \omega$

$$= \sum_{i=1}^{\infty}\int \rho_i \Big(\sum_{j=1}^{\infty} \rho'_j\Big)\omega$$

$$= \sum_{j=1}^{\infty}\int \rho'_j \Big(\sum_{i=1}^{\infty}\rho_i\Big)\omega = \sum_{j=1}^{\infty}\rho'_j\omega .$$

We can commute $\int$ and Σ because $\mathrm{supp}(\omega)$ is compact. $\square$

In particular. $\int_X \omega_1+\omega_2 = \int_X\omega_1 + \int_X \omega_2$; $\int_X c\cdot\omega = c\cdot\int_X \omega$.

· If the orientation is reversed, $\int_X \omega \rightsquigarrow -\int_X \omega$.
· After choosing a Riemannian metric, can define
$$vol(X) = \int_X \sigma_{vol}.$$

Thm. (Change-of-variable formula)
Let X', X be oriented n-mfolds and $f: X' \to X$ be an orientation-preserving diffeomorphism. Given $\omega \in \Omega_c^n(X)$, we have
$$\int_{\cancel{X}X'} \phi^* \omega = \int_X \omega.$$

proof. Given $\phi_0: U_0 \to U \subseteq X$ parametrization,
$f^{-1} \circ \phi_0 : U_0 \to f^{-1}(U) \subseteq X'$ is a parametrization.

Then $\int_{f^{-1}(U)} \cancel{\phi} f^* \omega = \int_{U_0} (f^{-1} \circ \phi_0)^* f^* \omega = \int_{U_0} \phi_0^* \omega = \int_U \omega$.

In general, given an covering $\{U_\alpha\}_{\alpha \in I}$ of X,
$\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X'.
If $\{\rho_i\}_{i \ge 1}$ is a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in I}$,
so is $\{\rho_i \circ f\}$.
So, $\int_X \omega = \sum_{i=1}^{\infty} \int_X \rho_i \omega = \sum_{i=1}^{\infty} \int_{U_i} \rho_i \omega$
$$= \sum_{i=1}^{\infty} \int_{f^{-1}(U_i)} (\rho_i \circ f) f^* \omega = \cancel{\sum} \int_{X'} f^* \omega. \square$$

Thm. If $\mu \in \Omega_c^{n-1}(X)$, then $\int_X d\mu = 0$.
proof. w.l.o.g. we can assume that $supp(\mu) \subseteq U$ parametrizable.
Then this follows from the result on $\mathbb{R}^n$. $\square$

§ 4.6 Stokes' theorem
$X \Rightarrow$ oriented ~~mf~~ manifold. $D \subseteq X$ domain.