§4.3. Vector fields & differential forms on mflds.

Def^n. Let X be a mfld. A <u>vector field</u> on X, written as $v \in \mathfrak{X}(X)$, is a function which assigns to each $p \in X$ an element $v(p) \in T_pX$.

A k-form ω is a function which assigns to each $p \in X$ an element $\omega(p) \in \Lambda^k(T_p^*X)$. $\quad$ ($C^\infty \Rightarrow \Omega^k(X)$)

Let X, Y be mflds, $f: X \to Y$ a C^∞-mapping.

Def^n. (A). Given $v \in \mathfrak{X}(X)$, $\omega \in \mathfrak{X}(Y)$, they are called $\boxed{f\text{-related}}$ iff $\forall p \in X$, we have $(df)_p v(p) = \omega(f(p))$.

~~The vector field defined by (f_*v)~~

For f being a diffeomorphism, $\omega(q) = (df)_{f^{-1}(q)} v_{f^{-1}(q)}$,

we write $\omega = f_* v \Rightarrow$ <u>push forward</u>.

The reversed direction: $(f^{-1})_* \omega$ is called <u>pullback</u>.

(B) Given $\omega \in \Omega^k(Y)$, its pullback $f^*\omega \in \Omega^k(X)$ is defined by

$$(f^*\omega)_p := (df_p)^*(\omega_{f(p)}).$$

Prop^n.

- Given mflds X, Y, Z and $X \xrightarrow{f} Y \xrightarrow{g} Z$ C^∞-mappings, for $\omega \in \Omega^k(Z)$, we have $(g \circ f)^*\omega = f^*(g^*\omega)$.
- If f, g are diffeomorphisms, then for $v \in \mathfrak{X}(X)$, we have ~~fo~~ $(g \circ f)_*(v) = g_*(f_*(v))$.

proof. Reduced to open subsets in $\mathbb{R}^n$. $\square$

Def^n. $X \Rightarrow n$-dim mfld, $U \subseteq X$ open is called <u>parametrizable</u> if $\exists\, U_0 \subseteq \mathbb{R}^n$ open & $\phi_0: U_0 \xrightarrow{\sim} U$ diffeomorphism.

Def^n. A k-form ω on ~~X~~ U is called smooth if $\phi_0^*\omega$ is smooth.

Lemma. Smoothness is well-defined.

proof. Given

$$\begin{tikzcd} U_0 \arrow[r, "\phi_0"] \arrow[d, "\phi_1^{-1}\circ\phi_0"'] & U \\ U_1 \arrow[ur, "\phi_1"'] & \end{tikzcd}$$

, then map $\psi := \phi_1^{-1} \circ \phi_0$ is a diffeomorphism.

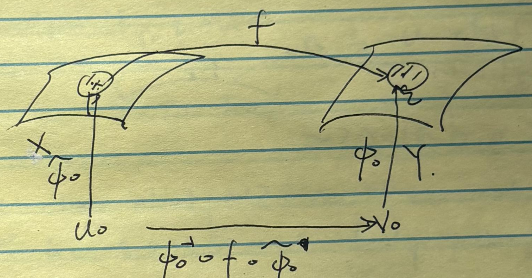
So, $\phi_0^*\omega = (\phi_1 \circ \phi_1^{-1} \circ \phi_0)^*\omega = (\phi_1^{-1}\circ\phi_0)^*\phi_1^*\omega$ $\Rightarrow \phi_0^*\omega$ is C^∞ iff $\phi_1^*\omega$ is C^∞. $\square$

Def^n. A vector field v on X is C^∞ iff $\phi_0^* v$ is C^∞ $\Rightarrow$ well-defined.

Def^n. A k-form ω on X is C^∞ iff $\forall p \in X$, $\exists$ ~~a~~ $\underbrace{\text{a}}_{\text{open}}$ neighborhood of p on which ω is C^∞. Similar for vector fields.

~~Pp~~ Prop^n. Given $\omega \in \Omega^k(Y)$, $f^*\omega$ is also C^∞ if f is smooth.

proof.



Choose parametrizable open neighborhoods as above $\Rightarrow \phi_0^{-1} \circ f \circ \tilde{\phi}_0$ is C^∞.

This shows that ~~$(\phi_0 \ldots$~~

$$\tilde{\phi}_0^* f^* \omega = \tilde{\phi}_0^* f^* (\phi_0^{-1})^* \phi_0^* \omega$$
$$= (\phi_0^{-1} \circ f \circ \tilde{\phi}_0)^* \phi_0^* \omega \text{ is } C^\infty. \quad \square$$

Rmk. If $f: X \to Y$ is a diffeomorphism, & v is a C^∞-vector field on X, $f_* v$ is also C^∞. $\square$

The following are the analogues of statements over $\mathbb{R}^n$.

- support: $\begin{cases} \text{for } \omega \in \Omega^k(X), \ \operatorname{supp}(\omega) := \overline{\{p \in X \mid \omega(p) \neq 0\}}. \\ \text{for } v \in \mathfrak{X}(X), \ \operatorname{supp}(v) := \overline{\{p \in X \mid v(p) \neq 0\}}. \end{cases}$
- integral curve: $I \subseteq \mathbb{R}$ open interval, $\gamma: I \to X$ C^∞ curve.

  Let $\vec{u} := (t_0, 1) \in T_{t_0}\mathbb{R}$ be the unit vector.

  For $v \in \mathfrak{X}(X)$, γ is called an integral curve iff

$$d\gamma_{t_0}(\vec{u}) = v_{\gamma(t_0)} \in T_{\gamma(t_0)} X.$$

Prop^n. If $v \in \mathfrak{X}(X)$, $w \in \mathfrak{X}(Y)$ are f-related, then an integral curve γ of v is mapped to an integral curve of w, $f \circ \gamma$.

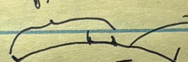
proof. ~~$(f \circ \gamma)_*$~~ $d(f \circ \gamma)_{t_0}(\vec{u}) = (df)_{\gamma(t_0)} (d\gamma)_{t_0}(\vec{u}) = (df)_{\gamma(t_0)} (v_{\gamma(t_0)})$
$$= w_{f \circ \gamma(t_0)}. \quad \square$$

~~Local existence: $\forall p$~~ For a vector field $v \in \mathfrak{X}(X)$,

Prop^n. Given a parametrizable $U \subseteq X$, we have

① <u>Local existence</u>: $\forall p \in U$, $\exists$ an integral curve $\gamma(t)$ of v defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and $\gamma(0) = p$.

② <u>Local uniqueness</u>: $\gamma_1 : I_1 \to U$, $\gamma_2 : I_2 \to U$ integral curves



I_1 I_2

w. $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$

$\Rightarrow \gamma_1 \cup \gamma_2|_{I_1 \cup I_2}$ is also an integral curve.

③ $\forall p \in U$, $\exists$ open nbhd $p \in O \subseteq U$ ~~and~~, $\varepsilon > 0$, and C^∞ $h : (-\varepsilon, \varepsilon) \times O \to U$ s.t. $\forall p$ $\gamma_p(t) := h(t, p)$ is an integral curve s.t. $\gamma_p(0) = p$.

proof. Follows from the discussion in $\mathbb{R}^n$. $\square$

Recall. v is called complete if $\forall p \in X$ $\exists$ an integral curve w. $\gamma(0) = p$ which exists in ~~all t~~ all time.

Thm. If X is compact, or more generally, $\mathrm{supp}(v)$ is compact, then v is complete.

proof. Suppose $\mathrm{supp}(v)$ is compact.

Given $p \in X$, by local existence, $\exists$ $\varepsilon > 0$ s.t. we can find an integral curve

$\gamma : (-\varepsilon, \varepsilon) \to X$ s.t. $\gamma(0) = p$.

Because $\mathrm{supp}(v)$ is compact, we know that $\forall \{x_k\}_{k \geq 1}$ s.t. $x_k \xrightarrow{k \to \infty} \pm\varepsilon$, the set $\{\gamma(x_k)\}_{k \geq 1}$ is bounded, also Cauchy in X.

By comparing arbitrarily two sequences, we can uniquely extend γ to a continuous map $\gamma : [-\varepsilon, \varepsilon] \to X$.

Using local existence again, at $\gamma(\pm\varepsilon)$, we can ~~extend~~ find integral curves ~~[illegible]~~

$$\begin{cases} \gamma_\varepsilon : (\varepsilon - \delta_1, \varepsilon + \delta_1) \to X \\ \gamma_{-\varepsilon} : (-\varepsilon - \delta_2, -\varepsilon + \delta_2) \to X. \end{cases}$$

Then the union $\gamma_{-\varepsilon} \cup \gamma \cup \gamma_\varepsilon$ is continuous, actually C^1 (thus C^∞ by ODE theory), so we have extended the solution to a bigger interval.

This implies that the solution exists for all time. $\square$

Therefore. $f_t \Rightarrow$ 1-parameter group of diffeomorphisms.

__Def^{n}__. (Exterior derivative).

$$\phi_0 : U_0 \longrightarrow U \subseteq X \text{ parametrization.}$$

Given $\omega \in \Omega^k(U)$, define

$$d\omega := (\phi_0^{-1})^* d(\phi_0^* \omega).$$

__Lemma__. This is well-defined.

__proof__. For
$$\begin{tikzcd} U_0 \arrow[r, "\phi_0"] \arrow[d, "\phi_1^{-1}\circ\phi_0"'] & U \\ U_1 \arrow[ur, "\phi_1"'] & \end{tikzcd},$$

we have $(\phi_0^{-1})^* d(\phi_0^* \omega)$

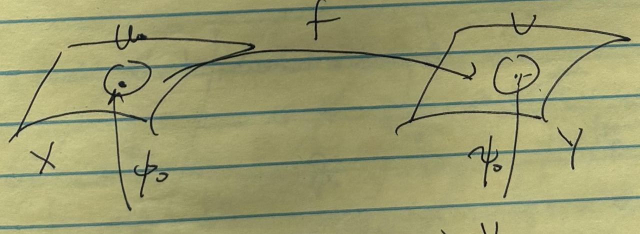
$$= (\phi_1^{-1})^* \cdot (\phi_0^{-1} \cdot \phi_1)^* d(\phi_0^* \omega)$$
$$= (\phi_1^{-1})^* d((\phi_1^*) \cdot (\phi_0^{-1})^* \cdot (\phi_0)^* \omega)$$
$$= (\phi_1^{-1})^* d(\phi_1^* \omega). \ \square$$

Globally, we defined $d\omega$ on ~~of~~ parametrizable open nbhds of points.

__Thm__. Given $f: X \to Y$ C^∞ and $\omega \in \Omega^k(Y)$, we have

$$f^*(d\omega) = d f^* \omega.$$

__proof__.



$$U_0 \xrightarrow[\psi_0^{-1} \circ f \circ \phi_0]{} V_0$$

Choose parametrizable open nbhds of $p \in U_0$, $\underset{f(p)}{\overset{\shortparallel}{q}} \in V_0$ s.t. $f(U_0) \subseteq V_0$,

$$f^*(d\omega) = f^*((\psi_0^{-1})^* d\, \psi_0^* \omega)$$
$$= (\phi_0^{-1})^* \underbrace{\phi_0^* f^* (\psi_0^{-1})^*}\, d\, \psi_0^* \omega$$
$$= (\phi_0^{-1})^* d\, \phi_0^* f^* (\psi_0^{-1})^* \psi_0^* \omega = (\phi_0^{-1})^* d\, \phi_0^* f^* \omega$$
$$= d(f^* \omega). \ \square$$

· Finally: one can define the interior product $\iota_v \omega$ via local parametrization,

and we have:

Thm. (Cartan's formula). $L_v \omega = d\iota_v \omega + \iota_v d\omega$,

where $L_v \omega = \frac{d}{dt}\Big|_{t=0} f_t^* \omega$ where f_t is the one-parameter group associated w/ v.