§3.6. Degree theory: application & computation.

Recall. $U, V \subseteq \mathbb{R}^n$ open subset

$f: U \to V$ C^∞ map, proper.

Choosing $\omega_0 \in \Omega_c^n(V)$ s.t. $\int_V \omega_0 = 1$, $\deg(f) = \int_U f^* \omega_0$.

Goal. Prove that $\deg(f)$ "=" $\# f^{-1}(y)$.

Def^n. $x \in U$ is called a critical point of f iff $\det(Df(x)) = 0$.

$\Leftrightarrow Df(x)$ is singular.

$\mathrm{Crit}(f)$. Its image called critical $\boxed{\text{values}}$. $f(\mathrm{Crit}(f))$.

The set $\mathrm{Im}(f) \setminus f(\mathrm{Crit}(f)) \Leftrightarrow$ set of regular values. $\mathrm{Reg}(f)$.

Convention. If $y \in V \setminus \mathrm{Im}(f)$, it's also called a regular value.

Thm. (Sard) $\mathrm{Reg}(f)$ is open and dense in V.

openness is straightforward; density will be proved later.

Thm. ~~If $q \in \mathrm{Reg}(f)$, then f~~

$f: U \to V$, C^∞ map, proper. If $q \in \mathrm{Reg}(f)$, then $f^{-1}(q)$ is finite.

If writing $f^{-1}(q) = \{p_1, \cdots, p_r\}$, then $\exists$ open nbhds of $p_i \in U_i$ in U s.t.

and $q \in W$ in V s.t.

(1) for $i \neq j$, U_i & U_j are disjoint

(2) $f^{-1}(W) = U_1 \cup \cdots \cup U_r$

(3) f maps U_j diffeomorphically onto W.

proof. If $p \in f^{-1}(q)$, then $Df(p)$ is non-singular.

By the inverse function theorem, $\exists\, p \in U_p$, $q \in V_q$ open

s.t. $f|_{U_p}: U_p \to V_q$ is a diffeomorphism.

In particular, $f^{-1}(q)$ is discrete: ~~any~~ $U_p \setminus \{p\}$ is not mapped to q.

By properness of f, $f^{-1}(q)$ is compact $\Rightarrow f^{-1}(q)$ is finite.

Take U_{p_j}, V_{p_j}, shrink them s.t. $U_{p_i} \cap U_{p_j} = \phi$ for $i \neq j$.

This can be done by finiteness + local diffeomorphism property.

Then take $W = \bigcap_{j=1}^{r} V_{p_j}$, and $U_j := (U_{p_j}) \cap f^{-1}\left(\bigcap_{j=1}^{r} V_{p_j}\right)$. $\square$

Thm. For each $p_i \in f^{-1}(q)$, let $\sigma_{p_i} := +1$ if $f: U_i \to W$ is orientation-preserving, $\sigma_{p_i} := -1$ if $f: U_i \to W$ is orientation reversing.

Then $\deg(f) = \sum_i \sigma_{p_i}$.

proof. Let W, ~~U_p~~ $U_1, \dots, U_r$ be as above.

Take $\omega_0 \in \Omega_c^n(W)$ s.t. $\int \omega_0 = 1$.

Then for $f|_{U_i}: U_i \to W$,

$$\deg(f|_{U_i}) = \int f^*\omega_0 = \begin{cases} 1 & \text{if } f \text{ is o.p.} \\ -1 & \text{if } f \text{ is o.r.} \end{cases}$$

So, $\deg(f) = \int f^*\omega_0 = \sum_j \int_{U_j} f^*\omega_0 = \sum_j \sigma_j$. □

Cor. If f is not surjective, then $\deg(f) = 0$.

proof. ~~Just~~ Just ~~take~~ take $q \in V$ s.t. $q \notin \mathrm{Im}(f)$. □

Alternatively: $V \setminus \mathrm{Im}(f)$ ~~contains~~ is open.

We can take $\omega_0 \in \Omega_c^n(V \setminus \mathrm{Im}(f))$ to compute $\deg(f)$. □

Cor. If $\deg(f) \neq 0$, then $f: U \twoheadrightarrow V$ is surjective.

Def^n. For $f_0, f_1: U \to V$ proper C^∞ maps, a proper homotopy is a C^∞ proper map $F: A \times U \to V$, where $A \subseteq \mathbb{R}$ is an open interval s.t. $F|_{\{0\} \times U} = f_0$, $F|_{\{1\} \times U} = f_1$.

Thm. If f_0 & f_1 are properly homotopic, then $\deg(f_0) = \deg(f_1)$.

proof. Consider the function

$$t \mapsto \int_U F_t^* \omega_0, \text{ where } \omega_0 \in \Omega_c^n(V) \text{ satisfies } \int \omega_0 = 1.$$

Then this is a continuous function taking discrete values.

So, it is a constant $\Rightarrow \deg(f_0) = \deg(f_1)$. □

Rmk. ① We realize that $\sum_i \sigma_{p_i}$ doesn't depend on q.

② There is a direct proof of homotopy invariance based on counting points.

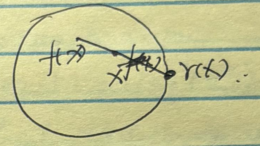
TWO APPLICATIONS

<u>Thm</u>. (Brouwer's fixed point theorem).

Let $B^n := \{x \in \mathbb{R} \mid |x| \le 1\}$.

If $f: B^n \to B^n$ is a continuous map, then f admits a fixed point.

<u>proof</u>. Consider the ray $s \mapsto f(x) + s(x - f(x))$, $s \in [0, +\infty)$



Let $r(x)$ be the intersection with the unit sphere $S^{n-1} \subseteq B^n$.

Then if $x \in S^{n-1}$, $r(x) = x$ by construction.

So we defined $r: B^n \to S^{n-1}$

s.t. $\forall x \in S^{n-1} = \partial B^{n-1}$, $r(x) = x$.

We can extend ~~to~~ r ~~to~~ be continuous function

$r: \mathbb{R}^n \to \mathbb{R}^n$

w/. ~~$r \equiv x$ out~~ $r(x) = x$ outside B^n $\Rightarrow$ $\mathrm{im}(r) = \mathbb{R}^n \setminus \mathring{B}^n$.

<u>Next</u>: (a) Mollifying $r \rightsquigarrow r_\varepsilon$, C^∞, $\|r_\varepsilon - r\|_{C^0} < \varepsilon$

(b) Partition of unity: $\rho: \mathbb{R}^n \to \mathbb{R}$, $\mathrm{supp}(\rho) \subseteq B$,

$\rho \geq 0$, ~~$\mathrm{supp}(\rho)$~~ $\rho = 1$ on $B_{1/3}$.

Then look at $\rho \cdot r_\varepsilon + (1-\rho) \cdot \mathrm{Id} = \varphi$

$\Rightarrow$ It's smooth, w/. image $\subseteq \mathbb{R}^n \setminus B_{1/3}$, $\equiv x$ outside B^n.

This is a contradiction: take $q \in \mathbb{R}^n \setminus B^n \Rightarrow \varphi^{-1}(q) = q$, $\deg(\varphi) = 1$;

take $q = 0 \in \mathbb{R}^n \Rightarrow \varphi^{-1}(0) = \phi$, $\deg(\varphi) = 0$. Σ □

<u>Thm</u>. (Fundamental thm of algebra).

The complex polynomial in one variable

$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ must have n solutions

(counted w/. multiplicity).

<u>proof</u>. <u>Claim</u>. The homotopy $t \mapsto z^n + t \cdot (a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$ is proper.

Point: for $|z| >> 1$, we have

$\frac{1}{2}|z|^n \geq |a_{n-1}| \cdot |z|^{n-1} + \dots + |a_1| \cdot |z| + |a_0|$.

$\Rightarrow |f(z)| \leq C$ is contained in

$\left| |z|^n - (|a_{n-1}||z|^{n-1} + \dots + |a_1| \cdot |z| + |a_0|) \right| \leq C$

$\Rightarrow$ is contained in $\frac{1}{2}|z|^n \leq C$

$\Rightarrow$ This is a proper homotopy.

As a result, $\deg(f) = \deg(z^n) = n$. $\square$

Rmk. One can do a local calculation to deal w/ multiplicity. $\square$

Proof of Sard's Theorem (following Milnor).

Thm. Let $f: U \to V$ be a smooth map, let C be the set of critical points.
$U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^p$ (not full rank, $\operatorname{rank}(df_x) < p$).

Then $f(C)$ has measure zero.

proof. We assume that $p \geq 1$ and perform an induction on n.

For $n=0$, the statement holds trivially.

Define $C_1 := \{x \in U \mid Df(x) = 0\}$.

More generally, $C_k := \left\{x \in U \;\middle|\; \frac{\partial^j f}{\partial x_{i_1} \cdots \partial x_{i_j}} = 0, \ \forall\, 1 \leq j \leq k,\ i_1, \dots, i_j \in \{1, \dots, n\}\right\}$.

Then we have

$$C \supset C_1 \supset C_2 \supset \cdots \supset C_k \supset \cdots.$$

Step 1. $f(C \setminus C_1)$ has measure zero.

Recall (Fubini's theorem). A measurable set $A \subseteq \mathbb{R} \times \mathbb{R}^{p-1}$ has measure 0
if $\forall c \in \mathbb{R}$, $A \cap (\{c\} \times \mathbb{R}^{p-1})$ has measure zero.

To prove $f(C \setminus C_1)$ has measure zero, we show that $\forall\, p \in C \setminus C_1$ ~~$q \in f(C \setminus C_1)$~~,
$\exists$ open nbhd $p \in V$ s.t. $f((C \setminus C_1) \cap V)$ has measure zero.

By covering $f(C \setminus C_1)$ using countably many such V, we can conclude.

Given $\bar{x} \in C \setminus C_1$, w.l.o.g. we assume that $\frac{\partial f_1(\bar{x})}{\partial x_1} \neq 0$, where $f = (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n))$.

Consider $h(x) = (f_1(x), x_2, \dots, x_n)$, which defines a ~~of~~ diffeomorphism

$$h : V \to V' \text{ where } \bar{x} \in V.$$

Then for $g = f \circ h^{-1} : V' \to \mathbb{R}^p$, we have $\mathrm{Crit}(g) = h(\mathrm{Crit}(f) \cap V)$,

and $g(\mathrm{Crit}(g)) = f(\mathrm{Crit}(f) \cap V)$.

It suffices to show that $g(\mathrm{Crit}(g))$ has measure zero.

Observe that $g(x_1, \cdots, x_n) = (f_1(h^{-1}(x)), \cdots, f_n(h^{-1}(x)))$

$(t, x_2, \cdots, x_n)$ $\qquad = (x_1, g_2(x_1, \cdots, x_n), \cdots, g_n(x_1, \cdots, x_n))$.

So, $x \in \mathrm{Crit}(g)$ iff $(x_2, \cdots, x_n)$ is a critical point of the mapping

$$(x_2, \cdots, x_n) \longmapsto (g_2(\overset{t}{\cancel{x_1}}, x_2, \cdots, x_n), \cdots, g_n(\overset{t}{\cancel{x_1}}, x_2, \cdots, x_n)) = \tilde{g}_t(x_2, \cdots, x_n)$$

Using the inductive assumption, we know that $\forall t$,

$$\tilde{g}_t(\mathrm{Crit}(\tilde{g}_t)) \text{ has measure zero.}$$

This means that $\forall t$, $g(\mathrm{Crit}(g) \cap (\{t\} \times \mathbb{R}^{n-1}))$ has measure zero.

By Fubini's theorem, Step 1 is done.

Step 2 $\quad \forall k \geq 1$, $f(C_k \setminus C_{k+1})$ has measure 0.

Take $\bar{x} \in C_k \setminus C_{k+1}$, then $\exists$

$$\frac{\partial^{k+1} f_r}{\partial x_{i_1} \cdots \partial x_{i_{k+1}}}(\bar{x}) \neq 0.$$

Let $w(x) = \dfrac{\partial^k f_r}{\partial x_{i_2} \cdots \partial x_{i_{k+1}}}$, and w.l.o.g., assume that $i_1 = 1$.

Consider $h(x) = (w(x), x_2, \cdots, x_n)$,

which defines a diffeomorphism over some open nbhd $\bar{x} \in V$.

$\Downarrow$ $\qquad\qquad V \xrightarrow{h} V'$ (where V is S)

AND:

As in Step 1, we can look at $\qquad h(V \cap C_k) \subseteq \{0\} \times \mathbb{R}^{n-1}$

$\qquad g = f \circ h^{-1} : V' \to \mathbb{R}^p$ $\qquad\qquad$ ↓ kth derivative.

~~and consider the slices $\mathrm{Im}(g) \cap$~~

and its restriction to $(\{0\} \times \mathbb{R}^{n-1}) \cap V'$, call it $\bar{g}$.

Then $\bar{g}$ is a function in $(n-1)$ variables, ~~and it has~~

Each point in $h(C_k \cap V)$ is a critical point of $\bar{g}$ $\Rightarrow$ $\bar{g}\, h(C_k \cap V) = f(C_k \cap V)$

has measure zero. $\Rightarrow$ By a countable cover, we're done w/ Step 2.

Step 3 For $k \gg 1$, $f(C_k)$ has measure zero.

Let $I^n \subset U$ be a cube of side length δ,

we show that $f(C_k \cap I^n)$ has measure 0.

Then because we can use countably many I^n covering C_k, we are done.

If $x \in C_k$, using Taylor expansion, we know that

$$|f(x+h) - f(x)| \le C \cdot |h|^{k+1} \quad (*)$$

if h lies in a cube I^n of side length δ.

Now subdivide I^n into r^n cubes, each of which has side length $\frac{\delta}{r}$.

If $x \in C_k \cap I_r^n$, where I_r^n is one of those cubes,

any point in the cube can be written as $x+h$ w/ $|h| \le \sqrt{n}\frac{\delta}{r}$.

By $(*)$, $f(I_r^n)$ lies in a cube of side length $\frac{C\delta^{k+1} n^{\frac{k+1}{2}}}{r^{k+1}}$,

so $f(C_k \cap I^n)$ has measure at most

$$r^n \cdot \left(\frac{C\delta^{k+1} n^{\frac{k+1}{2}}}{r^{k+1}}\right)^p.$$

As long as $p(k+1) > n \Rightarrow k > \frac{n}{p} - 1$, by taking $r \to \infty$,

we see that $\mu(f(C_k \cap I^n)) = 0$ $\square$