§2.5, 2.6 Interior product & pullback

Def^n. $U \subseteq \mathbb{R}^n$ open subset

Given $v \in \mathfrak{X}(U)$, $\omega \in \Omega^k(U)$,

the interior product $\iota_v \omega$ is a $(k-1)$-form satisfying

$$(\iota_v \omega)_p = \iota_{v_p} \omega_p. \to \text{pointwise interior product}$$

Prop^n. (1) <u>Linearity</u>: $\iota_v(\lambda_1 \omega_1 + \lambda_2 \omega_2) = \lambda_1 \iota_v \omega_1 + \lambda_2 \iota_v \omega_2$

$$\iota_{(\mu_1 v_1 + \mu_2 v_2)} \omega = \mu_1 \iota_{v_1} \omega + \mu_2 \iota_{v_2} \omega.$$

(2) <u>Derivation property</u>: for $\omega \in \Omega^k(U)$, $\mu \in \Omega^\ell(U)$,

$$\iota_v (\omega \wedge \mu) = (\iota_v \omega) \wedge \mu + (-1)^k \omega \wedge \iota_v \mu.$$

(3) ~~toto~~ $\iota_{v_1} \iota_{v_2} \omega = -\iota_{v_2} \iota_{v_1} \omega$. $\rightsquigarrow$ $\iota_v \iota_v \omega = 0$.

(4) Formula for decomposable k-forms: given $\mu_i \in \Omega^1(U)$, $1 \le i \le k$,

$$\iota_v(\mu_1 \wedge \cdots \wedge \mu_k) = \sum_{r=1}^{k} (-1)^{r-1} \cancel{\mu_i}\, \iota_v \mu_r \cdot \mu_1 \wedge \cdots \wedge \hat{\mu}_r \wedge \cdots \wedge \mu_k.$$

Example. If $v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$,

then $\iota_v \omega = \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Rmk. Different from the textbook, we define Lie derivatives via pullback and prove Cartan's formula.

Given $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open subsets, and $f: U \to V$ a C^∞-map,

the linear map $df_p: T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$

induces $df_p^* := (df_p)^*: \Lambda^k(T^*_{f(p)}\mathbb{R}^m) \to \Lambda^k(T^*_p\mathbb{R}^n)$

Def^n. Given $\omega \in \Omega^k(V)$, its $\boxed{\text{pullback}}$ under f

$f^*\omega$ is the k-form on U which assigns to $p \in U$

the value ~~$(df_p)^* \omega_p$~~. $df_p^*(\omega_{f(p)})$.

Prop^n. (1) If $\phi \in \Omega^0(V)$, i.e. $\phi: V \to \mathbb{R}$ is a smooth function,

then $f^*\phi$ is the composition $\phi \circ f$.

(2) Continuing (1), write $\mu = d\phi$, then

$$f^*\mu = f^* d\phi = d(f \circ \phi) = d f^* \phi \rightsquigarrow \text{chain rule.}$$

(3) $f^*(\lambda_1 \omega_1 + \lambda_2 \omega_2) = \lambda_1 f^* \omega_1 + \lambda_2 f^* \omega_2$.

(4) f^* is an algebra map.

$$f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2) \Rightarrow \text{true pointwisely.}$$

(5) Let $W \subseteq \mathbb{R}^l$ be an open subset and suppose $g: V \to W$ is a C^∞-map.

Given $p \in U$, $q = f(p)$, and ~~the~~ $r = g(q)$, then

the composition of $(df_p)^*: \Lambda^k(T_q\mathbb{R}^m) \to \Lambda^k(T_p\mathbb{R}^n)$

and $(dg_q)^*: \Lambda^k(T_r\mathbb{R}^l) \to \Lambda^k(T_q\mathbb{R}^m)$

is $(dg_q \circ df_p)^* = (dg_q)^* \circ (df_p)^*$.

By chain rule, $dg_q \circ df_p = d(f \circ g)_p$.

So we have $f^*(g^*\omega) = (g \circ f)^*\omega$.

(6) In local coordinates, if $\omega = \sum_I \phi_I dx_I$,

then $f^*\omega = \sum_I f^*\phi_I \, f^*dx_I$.

Here if $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, then $f^*dx_I = df_{i_1} \wedge \cdots \wedge df_{i_k}$.

Prop^n. $f^*d\omega = df^*\omega$.

proof. Write down in ~~local~~ coordinates:

$$d\omega = \sum_I d\phi_I \wedge dx_I.$$

use the algebra property, $f^*d\omega = \sum_I f^*(d\phi_I) \wedge (f^*dx_I)$

use $f^*d\phi = d(f^*\phi)$ $= \sum_I d(f^*\phi_I) \wedge (f^*dx_I)$

$$= d\left(\sum_I f^*\phi_I \wedge f^*dx_I\right) = df^*\omega. \square$$

Suppose $\omega \in \Omega^k(U)$, $v \in \mathfrak{X}(U)$.

Let f_t be the one-parameter group of diffeomorphisms associated with v.

Def^n. The Lie derivative is defined to be

$$L_v\omega := \frac{d}{dt}\Big|_{t=0}(f_t^*\omega).$$

Morally: differentiate ω along the direction of v.

Thm (Cartan's formula). $L_v\omega = d\iota_v\omega + \iota_v d\omega$.

proof. Verification in standard coordinates.

Write $\omega = \sum_I \phi_I dx_I$, ~~then~~ $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$

$$f_t^*\omega = \sum_I \phi_I \circ f_t \, (df_t)_{i_1} \wedge \cdots \wedge (df_t)_{i_k}.$$

Differentiate w.r.t. ~~to~~ t and apply the chain rule + Leibniz rule, we have

$$\frac{d}{dt}\Big|_{t=0} f_t^* \omega = \sum_I \frac{d}{dt}\Big|_{t=0} (\phi_I \circ f_t)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$+ \sum_I \sum_{r=1}^{k} \phi_I (-1)^{r-1} \frac{d}{dt}\Big|_{t=0} \cancel{d(x_{i_r} \circ f_t)}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}.$$

$$= \sum_I d\phi_I(v)\, dx_I + \sum_I \phi_I \sum_{r=1}^{k} (-1)^{r-1}\, dv_{i_r}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}.$$

On the other hand,

$$d\iota_v \omega + \iota_v d\omega = d\Big(\sum_I \phi_I \sum_{r=1}^{k} (-1)^{r-1} v_{i_r}\, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}\Big)$$

$$= \cancel{d\big(\textstyle\sum_{r=1}}$$

$$+ \iota_v\Big(\sum_I d\phi_I \wedge dx_I\Big)$$

$$= \sum_I \sum_{r=1}^{k} \cancel{d\phi_I} (-1)^{r-1} v_{i_r}\, d\phi_I \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}$$

this is $\sum_I d\phi_I \wedge (\iota_v dx_I)$

$$+ \sum_I \sum_{r=1}^{k} (-1)^{r-1} \phi_I\, dv_{i_r} \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}$$

cancel.

$$+ \sum_I d\phi_I(v)\, dx_I - \sum_I d\phi_I \wedge \iota_v dx_I$$

$$= \sum_I d\phi_I(v)\, dx_I + \sum_I \phi_I \sum_{r=1}^{k} (-1)^{r-1} dv_{i_r} \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_r}} \wedge \cdots \wedge dx_{i_k}.$$

Therefore, $L_v \omega = d\iota_v \omega + \iota_v d\omega$. $\square$

Rmk. If $\omega \in \Omega^0(U)$, i.e. $\omega = f : U \to \mathbb{R}$ is a smooth function, then

$$L_v f = \frac{d}{dt}\Big|_{t=0} f \circ f_t = f_*(v). \ \square$$

Prop^n. (1) Lie derivative commutes w/ differentiation:

$$d L_v \omega = L_v d\omega.$$

(2) Lie derivative is compatible w/ wedge product

$$L_v(\omega \wedge \mu) = (L_v \omega) \wedge \mu + \omega \wedge (L_v \mu).$$

proof. (1) $dL_v \omega = d(\iota_v d\omega + d(\iota_v \omega)) = d\iota_v d\omega$ (because $d^2 = 0$).

$$L_v d\omega = (d\iota_v + \iota_v d)\, d\omega = d\iota_v d\omega.$$

(2) $L_v(\omega\wedge\mu) = (d\iota_v + \iota_v d)(\omega\wedge\mu)$

$$= d\left((\iota_v\omega)\wedge\mu + (-1)^k\omega\wedge(\iota_v\mu)\right)$$
$$+ \iota_v\left(d\omega\wedge\mu + (-1)^k\omega\wedge d\mu\right)$$
$$= (d(\iota_v\omega))\wedge\mu + (-1)^{k-1}\cancel{(\iota_v\omega)}\wedge d\mu + (-1)^k \cancel{d\omega}\wedge(\iota_v\mu) + (-1)^{2k}\omega\wedge d(\iota_v\mu)$$
$$+ (\iota_v d\omega)\wedge\mu + (-1)^{k+1}\cancel{d\omega}\wedge\iota_v\mu + (-1)^k\cancel{\iota_v\omega}\wedge d\mu + (-1)^{2k}\omega\wedge\iota_v d\mu$$
$$= L_v\omega\wedge\mu + \omega\wedge L_v\mu.$$

Alternatively. (1) $d(f_t^*\omega) = f_t^* d\omega$.

Take $\frac{d}{dt}\big|_{t=0}$ on both sides we obtain (1).

(2) $f_t^*(\omega\wedge\mu) = f_t^*\omega\wedge f_t^*\mu \rightsquigarrow$ take $\frac{d}{dt}\big|_{t=0}$. □

Example. For $U\subseteq\mathbb{R}^n$, write $v=\sum_{i=1}^n g_i\frac{\partial}{\partial x_i}$, then

$$L_v(dx_1\wedge\cdots\wedge dx_n) = \sum_{i=1}^n\left(\frac{\partial g_i}{\partial x_i}\right)dx_1\wedge\cdots\wedge dx_n.$$

$\boxed{\text{Discussion}}$ Homotopy invariance of pullback.

Def^n Given smooth maps $f_0, f_1 : U\to V$

a homotopy between f_0 and f_1 is a smooth map

$$F: U\times A\to V$$

where $A\subseteq\mathbb{R}$ is an open interval containing 0 and 1 such that

$$F|_{U\times\{0\}} = f_0,\quad F|_{U\times\{1\}} = f_1.$$

Thm If f_0 and f_1 are homotopic, then for $\omega\in\Omega^k(V)$ s.t. $d\omega=0$,

the form $f_0^*\omega - f_1^*\omega$ is exact. $\rightsquigarrow$ $\boxed{\text{exercise}}$

Cor. If U is contractible, then every closed k-form must be exact if $k\geq 1$.

§2.7 div, curl, and ∇.

Goal Use $d:\Omega^k\to\Omega^{k+1}$ and duality to understand operations on vector fields.

Recall Let $B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be a (bilinear) inner product.

Then B induces an isomorphism $\mathbb{R}^n \xrightarrow{\sim} (\mathbb{R}^n)^*$

$$v\longmapsto B(v,-).$$

For $\mathbb{R}^n$, we consider $((x_1,\dots,x_n),(y_1,\dots,y_n))\mapsto x_1y_1+\cdots+x_ny_n$.

Then we have $T_p\mathbb{R}^n \xrightarrow{\cong} T_p^*\mathbb{R}^n$.

Example. $\frac{\partial}{\partial x_i}$ is mapped to dx_i

$\rightsquigarrow$ $v = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is mapped to $\sum_{i=1}^n f_i dx_i$.

Def^n. For a smooth function $f: U \to \mathbb{R}$, its gradient ∇f is defined to be the dual of $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

$\Rightarrow \nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$. $\rightsquigarrow$ fastest direction of increasing

Rmk. ∇ depends on the choice of B.

Example. Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$.

For $v = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$, $\iota_v \Omega = \sum_{i=1}^n (-1)^{i+1} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$.

Then $d(\iota_v \Omega) = \sum_{i=1}^n (-1)^{i+1} \frac{\partial f_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$= \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\right) \Omega$.

The divergence of the vector field is

$$\operatorname{div}(v) := \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}.$$

Example. Restricting to $n=3$, $\operatorname{curl}(v)$ is the vector field satisfying

$$d(\text{Dual of } v) = \iota_{\operatorname{curl}(v)} \Omega.$$

More concretely, write $v = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3}$,

then Dual of $v = \sum_{i=1}^n f_i dx_i$,

so $dv = \frac{\partial f_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial f_1}{\partial x_3} dx_3 \wedge dx_1$

$+ \frac{\partial f_2}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial f_2}{\partial x_3} dx_3 \wedge dx_2$

$+ \frac{\partial f_3}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial f_3}{\partial x_2} dx_2 \wedge dx_3$

$= \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3 \wedge dx_1$

$+ \left(-\frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2}\right) dx_2 \wedge dx_3$.

$\Rightarrow \operatorname{curl}(v) = (\partial_1 f_2 - \partial_2 f_1)\partial_3 + (\partial_3 f_1 - \partial_1 f_3)\partial_2 + (\partial_2 f_3 - \partial_3 f_2)\partial_1$.

Summary. They more or less all come from d.

Maxwell Equation.

Let $\vec{v}_E, \vec{v}_M$ be the electric and magnetic fields respectively.
$q \Rightarrow$ scalar charge density, $\omega \Rightarrow$ current density, $c \Rightarrow$ velocity of light.
Then

$$\begin{cases} \mathrm{div}(\vec{v}_E) = q & ① \\ \mathrm{curl}(\vec{v}_E) = -\frac{\partial}{\partial t}\vec{v}_M & ② \\ \mathrm{div}(\vec{v}_M) = 0 & ③ \\ c^2\,\mathrm{curl}(\vec{v}_M) = \vec{\omega} + \frac{\partial}{\partial t}\vec{v}_E. & ④ \end{cases}$$

Using differential forms, write

$$\mu_E = \iota_{\vec{v}_E}\Omega, \quad \mu_M = \iota_{\vec{v}_M}\Omega.$$

Then $\boxed{d\mu_E = q\,\Omega}$, $\boxed{d\mu_M = 0}$

Define the "Hodge dual" as $(\alpha \mapsto *\alpha)$.

$1 \leftrightarrow dx_1 \wedge dx_2 \wedge dx_3$, $dx_1 \leftrightarrow dx_2 \wedge dx_3$, $dx_2 \leftrightarrow dx_3 \wedge dx_1$,
$dx_3 \leftrightarrow dx_1 \wedge dx_2$.

In other words, $\alpha \wedge *\alpha = |\alpha|^2 dx_1 \wedge dx_2 \wedge dx_3$. $*^2 = 1$.

Then $\mathrm{curl}(\vec{v}) = *d(*\mu)$ if $\mu = \iota_{\vec{v}}\Omega$.
the dual of

So ② & ④ become

$$*d*\mu_E = -*\frac{\partial}{\partial t}\mu_M, \quad c^2 * d * \mu_M = (\text{dual of } \vec{\omega}) + \frac{\partial}{\partial t} * \mu_E$$

$$\Rightarrow \boxed{d*\mu_E = -\partial_t \mu_M,} \boxed{c^2 d*\mu_M = *(\text{dual of } \vec{\omega}) + \cancel{*}\frac{\partial}{\partial t}\mu_E.}$$

Simplify things further, write (assuming $c=1$)

$$\omega_M = \mu_M - *\mu_E \wedge dt, \quad \omega_E = \mu_E - *\mu_M \wedge dt,$$

$$\Lambda := q\,\Omega + \iota_\omega \Omega \wedge dt,$$

then
$$d\omega_M = d\mu_M - d(*\mu_E \wedge dt)$$
$$= d_{\mathbb{R}^3}\mu_M - \partial_t \mu_M \wedge dt + \partial_t \mu_M \wedge dt = 0.$$

$$d\omega_E = d_{\mathbb{R}^3}\mu_E + dt \wedge \partial_t \mu_E - d(*\mu_M \wedge dt) = \Lambda.$$

Note: $\omega_E = *\omega_M = \omega$

$$\Rightarrow \begin{cases} d\omega = 0 \\ d^*\omega = 0. \end{cases}$$

Extra topics: symplectic form, Hamiltonian vector fields

§3. Integration of forms

Prop^n $U, V \subseteq \mathbb{R}^n$ open subset, $\overset{f}{\phi}: U \to V$ diffeomorphism.

Then for a continuous function $\phi: V \to \mathbb{R}$, we have

$$\int_V \phi(y)\, dy = \int_U (\phi \circ f)(x) |\det Df(x)|\, dx.$$

Goal. Prove this using differential forms.

§3.1, 3.2 Poincaré lemma for compactly supported n-forms.

Def^n. For $\omega \in \Omega^k(\mathbb{R}^n)$, its support is defined to be

$$\mathrm{supp}(\omega) := \overline{\{x \in \mathbb{R}^n \mid \omega_x \neq 0\}}.$$

ω is compactly supported $\iff$ $\mathrm{supp}(\omega)$ is compact.

Notation. $\Omega_c^k(\mathbb{R}^n)$, $\Omega_c^k(U)$.

Def^n. For $U \subseteq \mathbb{R}^n$, if $\omega \in \Omega_c^n(U)$ and write $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$

for $f \in C_c^\infty(U)$, define

$$\int_U \omega = \int_{\mathbb{R}^n} f\, dx.$$

- Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle.

Thm (Poincaré lemma). Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ s.t. $\mathrm{supp}(\omega) \subseteq \mathrm{int}(Q) = (a_1, b_1) \times \cdots \times (a_n, b_n)$.

The TFAE.

(1) $\int \omega = 0$

(2) $\exists\, \mu \in \Omega_c^n(\mathbb{R}^n)$ w/. $\mathrm{supp}(\mu) \subseteq \mathrm{int}(Q)$ s.t. $d\mu = \omega$

proof. (2) $\Rightarrow$ (1) This is integration by part.