Chapter 2. Differential forms on $\mathbb{R}^n$

§2.1 Vector fields and one-forms

<u>Def$^{\underline{n}}$</u>. Given $p \in \mathbb{R}^n$, the tangent space $T_p\mathbb{R}^n := \{(p,v) \mid v \in \mathbb{R}^n\}$.

Under the bijection $\begin{array}{l} T_p\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \\ (p,v) \mapsto v, \end{array}$ $T_p\mathbb{R}^n$ can be equipped w/. a vector space structure.

Given a C^1-map $f: U \to \mathbb{R}^m$ w/. $U \subseteq \mathbb{R}^n$ open,

if $q = f(p)$, the differential defines a linear map

$$df_p : T_p\mathbb{R}^n \longrightarrow T_q\mathbb{R}^m$$
$$(p,v) \mapsto (q, Df(p)v),$$

where $Df(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$, where $f = (f_1(x_1,\cdots,x_n), \cdots, f_m(x_1,\cdots,x_n))$.

If $\operatorname{im}(f) \subseteq V \subseteq \mathbb{R}^m$, and $g: V \to \mathbb{R}^k$ is a C^1-map, then

$$dg_q \circ df_p = d(f \circ g)_p \text{ by the chain rule.}$$

<u>Def$^{\underline{n}}$</u>. A <u>vector field</u> on $U \subseteq \mathbb{R}^n$ is a (continuous) function which
open
assigns to each point $p \in U$ a vector $v(p) \in T_p\mathbb{R}^n$.

<u>Rmk</u>. For $p_1 \neq p_2$, we view $T_{p_1}\mathbb{R}^n$ & $T_{p_2}\mathbb{R}^n$ as different spaces.

<u>Examples</u>: ① Given $v \in \mathbb{R}^n$, we can define a constant vector field

$$p \mapsto (p,v).$$

② Let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$,
the corresponding constant vector fields are written as $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$.

③ If $f: U \to \mathbb{R}$ is a function, we can define
fv whose value at p is $f(p)v(p)$

④ $v_1 + v_2 \Rightarrow$ pointwise summation

⑤ In general, v can be written as $\sum_{i=1}^n g_i(p)\frac{\partial}{\partial x_i}$.

v is called C^∞ if g_i's are C^∞.

$\underline{\text{Def}^{\underline{n}}}$. (Lie differentiation) Given $f \in C^1(U)$, define

$$L_v f := \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i} \text{ for a vector field } v = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}.$$

Exercise. Show that $L_v(f_1 \cdot f_2) = L_v(f_1) \cdot f_2 + f_1 \cdot L_v(f_2)$.

Dually, we can consider

$\underline{\text{Def}^{\underline{n}}}$. Given $p \in \mathbb{R}^n$, the cotangent space $T_p^* \mathbb{R}^n$ is defined to be

$$T_p^* \mathbb{R}^n := (T_p \mathbb{R}^n)^* . \rightarrow \text{cotangent vectors.}$$

$\underline{\text{Def}^{\underline{n}}}$. A differential 1-form ω on $U \subseteq \mathbb{R}^n$ (open) is a function which assigns to $p \in U$

a vector $\omega_p \in T_p^* \mathbb{R}^n$.

Examples. ① Given $f : U \to \mathbb{R}$, C^1, $df_p : T_p \mathbb{R}^n \to T_{f(p)} \mathbb{R}$.

Identifying $T_{f(p)} \mathbb{R} \cong \mathbb{R}$, we can view $df_p \in T_p^* \mathbb{R}^n$.

This defines a one-form, denoted by df.

② $\phi \cdot \omega$ for $\phi : U \to \mathbb{R}$; $\omega_1 + \omega_2$

③ Denote by $dx_1, \dots, dx_n$ the basis dual to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$,

then any 1-form ω can be written as

$$\omega = \sum_{i=1}^{n} f_i \, dx_i \text{ for } f_i : U \to \mathbb{R}.$$

ω is called C^∞ if f_i's are C^∞.

Exercise. Show that $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

$\underline{\text{Def}^{\underline{n}}}$. (Interior product) Given a vector field v and a 1-form ω,

$\iota_v \omega$ is the pairing between v and ω.

Example. For $\phi : U \to \mathbb{R}$ C^1, we have

$$\iota_v d\phi = L_v \phi \Rightarrow \text{differentiate along } v.$$

§2.2 Integrating vector fields

<u>Notation</u>. $U \subseteq \mathbb{R}^n$ open, $\mathfrak{X}(U) \Rightarrow$ space of vector fields on U.

<u>Def^n</u>. Given $v \in \mathfrak{X}(U)$, an integral curve of v is a C^1 map

$$\gamma : (a,b) \to U \text{ such that}$$

$$(\gamma, \dot\gamma) = (\gamma(t), \frac{d\gamma(t)}{dt}) = v(\gamma(t)).$$

<u>Facts about ODES</u>

<u>Thm</u>. For $v \in \mathfrak{X}(U)$, given $p_0 \in U$ and $a \in \mathbb{R}$, $\exists$ $T > 0$ and $p_0 \in U_0 \subseteq U$ s.t.
Existence $\forall p \in U$, we have an integral curve of v, $\gamma_p : (a-T, a+T) \to U$
with $\gamma_p(a) = p$.

<u>Thm</u> (Continuation + uniqueness) Given $v \in \mathfrak{X}(U)$ and integral curves of v

$$\gamma_1 : I_1 \to U, \quad \gamma_2 : I_2 \to U$$

such that $\gamma_1(a) = \gamma_2(a)$ for some $a \in I_1 \cap I_2$, then $\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$

and $\gamma : I_1 \cup I_2 \to U$ with $\gamma(t) := \begin{cases} \gamma_1(t), & t \in I_1 \\ \gamma_2(t), & t \in I_2 \end{cases}$ is also an integral curve.

<u>Thm</u>. (C^∞-dependence on initial data).

Given $v \in \mathfrak{X}(U)$, $V \subseteq U \subseteq \mathbb{R}^n$ open, $\overset{a}{\underset{}{I}} \subseteq \mathbb{R}$ an interval,

$h : V \times I \to U$ s.t. $\begin{cases} h(p,a) = p & (a \text{ is the initial time}) \\ h(p,-) : I \to U \text{ is an integral curve of } v, \end{cases}$

then h is C^∞.

<u>Exer</u>. If γ is an integral curve, so is $\gamma(\cdot + c)$ for any $c \in \mathbb{R}$.

<u>Def^n</u>. $\phi : U \to \mathbb{R}$ is an integral of $\dot\gamma = v$ if $\forall \gamma$ integral, $\phi(\gamma(t))$ is constant.

<u>Cor</u>. ϕ integral $\Leftrightarrow L_v \phi = 0$.

<u>proof</u>. $\frac{d}{dt}\phi(\gamma(t)) = d\phi \cdot \dot\gamma = d\phi \cdot v = L_v \phi$. $\square$

Def^n. A $v \in \mathfrak{X}(U)$ is called complete iff $\forall p \in U$, $\exists$ an integral curve starting @ p & $\exists$ for all time.

For an integral curve $\gamma: [0,b) \to U$, it is <u>maximal</u> iff one of the following happens,

① $b = +\infty$

② $|\gamma(t)| \to +\infty$ as $t \to b$

③ the limit set of $\{\gamma(t) \mid 0 \le t < b\}$ intersects ∂U.

Thm. If $\exists$ a proper and C^1-function $\phi: U \to \mathbb{R}$ s.t. $L_v \phi = 0$, then v is complete.

proof. It suffices to rule out ②, ③.

Because $\phi(\gamma(t)) = $ constant, ϕ is proper

$\Rightarrow$ $Im(\gamma)$ is compact, so ② & ③ cannot happen. $\square$

Def^n. The support of $v \in \mathfrak{X}(U)$ is defined as

$$supp(v) := \overline{\{q \in U \mid v(q) \neq 0\}}.$$

Thm. If $supp(v)$ is compact, then v is complete.

proof. If $supp(v)$ is compact, then $|v|$ is uniformly bounded.

By existence result of ODE, the solution $\exists$ for all time.

Outside $supp(v)$, any integral curve stays constant.

So, no integral curve can escape outside U. $\square$

Def^n. Given a complete vector field v, its associated 1-parameter group of C^∞-diffeomorphisms f_t is the map which assigns to $p \in U$ the value $\gamma(t)$,

where $\begin{cases} \gamma(0) = p \\ \dot{\gamma} = v. \end{cases}$

Properties

- $f_0 = Id$.
- $f_{t_1+t_2} = f_{t_1} \circ f_{t_2} \to$ exercise.

$\underline{\text{Def}^{\underline{n}}}$. Given $v \in \mathcal{X}(U)$ and $f: U \to W$ a C^∞-diffeomorphism, the pushforward $w = f_* v$ is uniquely specified by

$$w(f(p)) = df_p(v(p)).$$

More concretely: if $v = \sum v_j \frac{\partial}{\partial x_j}$, $f = (f_1(x_1, \cdots x_n), \cdots, f_n(x_1, \cdots, x_n))$,

$$\text{then } w_i(f(p)) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} v_j \text{ if } w = \sum w_i \frac{\partial}{\partial x_i}.$$

$$\Rightarrow w_i = \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} v_j\right) \circ f^{-1}.$$

Thm. If γ is an integral curve of v, then $f \circ \gamma$ is an integral curve of $f_* v$.

proof. $\frac{d}{dt}(f \circ \gamma) = df(\dot\gamma) = df(v) = f_* v$. $\square$

Cor. In the above setting, if $f_{1,t}$, $f_{2,t}$ are respectively the 1-parameter groups associated w/ v & $f_* v$, then $f \circ f_{1,t} = f_{2,t} \circ f$.

$\hookrightarrow$ $\boxed{\text{exercise}}$

Exercise: Show that $(f_2)_* \circ (f_1)_* = (f_2 \circ f_1)_*$.

$\boxed{\text{Duality}}$

$\underline{\text{Def}^{\underline{n}}}$. Given a 1-form μ on V and a C^1-map $f: U \to V$, the pullback $f^* \mu$ is defined by

$$\mu_{f(p)} \circ df_p : T_p\mathbb{R}^n \to \mathbb{R} \text{ for } p \in U.$$

Example. If $\mu = d\phi$, then $f^* \mu = d(\phi \circ f)$.

Exercise: If μ & f are C^∞, then $f^* \mu$ is C^∞.

§2.3 Differential k-forms

we only consider C^∞-forms

<u>Def^n</u>. For an open subset $U \subset \mathbb{R}^n$, a k-form ω on U
$\hookrightarrow \omega \in \Omega^k(U)$
is a function which assigns to each $p \in U$ an element $\omega_p \in \Lambda^k(T_p^* \mathbb{R}^n)$.

<u>Properties</u>: ① The pointwise wedge product defines the wedge product
$$\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

s.t. $(\omega_1 \wedge \omega_2)_p = (\omega_1)_p \wedge (\omega_2)_p$.

② In particular, $(df_1 \wedge \cdots \wedge df_k)_p = (df_1)_p \wedge \cdots \wedge (df_k)_p$ for C^1-functions $f_i : U \to \mathbb{R}$.

③ Note that $\Omega^k(U)$ admits a basis
$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \text{ where } i_1 < \cdots < i_k,$$
so each $\omega \in \Omega^k(U)$ can be uniquely written as
$$\omega = \sum_I f_I \, dx_I.$$

④ $\Omega^k(U)$ is a vector space and a module over $C^\infty(U)$, with $(f\omega)_p = f(p)\, \omega_p$ for $f \in C^\infty(U)$.

⑤ The differential of functions can be viewed as
$$d : \Omega^0(U) \to \Omega^1(U)$$
$$f \mapsto \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j, \text{ here } \Omega^0(U) = C^\infty(U).$$

§2.4. Exterior differentiation

<u>Goal</u>. Define $d : \Omega^k(U) \to \Omega^{k+1}(U)$ such that

ⓐ $d : \Omega^0(U) \to \Omega^1(U)$ agrees with the above ⑤.

ⓑ $d(\lambda \omega_1 + \mu \omega_2) = \lambda d\omega_1 + \mu d\omega_2$

ⓒ $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$ for $\omega_1 \in \Omega^k(U)$

ⓓ $d(d\omega) = 0$.

One consequence.

$$d(df_1\wedge\cdots\wedge df_k)$$
$$= (ddf_1)\wedge df_2\wedge\cdots\wedge df_k + (-1)\cdot df_1\wedge(ddf_2)\wedge\cdots\wedge df_k$$
$$+\cdots+(-1)^{k-1}df_1\wedge\cdots\wedge df_{k-1}\wedge(ddf_k)=0.$$

To construct d, we declare that

$$d\left(\sum_I f_I dx_I\right) = \sum_I df_I\wedge dx_I.$$

Thm. Such a d satisfies ⓐ – ⓓ.

proof. ⓐ obvious

ⓑ obvious.

ⓒ We can use the same formula to include I with repeating indices and not strictly increasing (just use signs of permutations to kill).

Then for $\omega_1=\sum f_I dx_I$, $\omega_2=\sum g_J dx_J$,

$$\omega_1\wedge\omega_2=\sum_{I,J} f_I g_J\, dx_I\wedge dx_J,$$

then
$$d(\omega_1\wedge\omega_2)=\sum_{I,J} g_J\, df_I\wedge dx_I\wedge dx_J+\sum_{I,J} f_I\, dg_J\, dx_I\wedge dx_J$$
$$= d\omega_1\wedge\omega_2+(-1)^{|I|}\sum_{I,J} f_I\, dx_I\wedge dg_J\wedge dx_J$$
$$= d\omega_1\wedge\omega_2+(-1)^k\omega_1\wedge d\omega_2.$$

ⓓ This is a direct calculation.

Given $\omega=\sum_I f_I dx_I$,

$$d\omega=\sum_I\sum_{i=1}^n\frac{\partial f_I}{\partial x_i}dx_i\wedge dx_I,$$
$$d(d\omega)=\sum_I\sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2 f_I}{\partial x_i\partial x_j}dx_j\wedge dx_i\wedge dx_I.$$

For $i=j$, $dx_i\wedge dx_j=0$.

For $i\neq j$, we can pair $\frac{\partial^2 f_I}{\partial x_i\partial x_j}dx_j\wedge dx_i$ with $\frac{\partial^2 f_I}{\partial x_j\partial x_i}dx_i\wedge dx_j$,

which cancel out. $\Rightarrow d^2=0$. $\square$

Def$^{\underline{n}}$. For $\omega \in \Omega^k(U)$, it's called $\boxed{\text{closed}}$ iff $d\omega = 0$.

$\boxed{\text{exact}}$ iff $\exists \mu \in \Omega^{k-1}(U)$ s.t. $d\mu = \omega$.

<u>In perticuler</u>: any exact k-form is closed.