Syllabus (textbook, lecture plans, HW, grade), website

§0. Introduction and Overview

Example 1. $U, V \subseteq \mathbb{R}^n$, open

$f: U \to V$ differentiable, bijective

$\phi: V \to \mathbb{R}$ bounded C^0-function

Then: $\int_V \phi \, dy = \int_U (\phi \circ f) \, |\det Jf| \, dx$

Here: Jf = Jacobi matrix, whose (i,j)-th entry $= \frac{\partial f_i}{\partial x_j}$,

where $f = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)) = (y_1, \dots, y_n)$.

For $n=1$, $\cancel{\int_V \phi \, dy = \int_U}$ $U = (a,b)$, $V = (c,d)$

$\int_c^d \phi \, dy = \int_a^b (\phi \circ f) \frac{df}{dx} dx$ "=" $\int_a^b f^*(\phi \, dy)$,

because "$f^* dy = d(f^* \circ y) = \frac{df}{dx} dx$".

Goal. Make sense of $\int_V \phi \, dy = \int_U f^*(\phi \, dy)$ in general, for manifolds.

Term: $\boxed{\text{change-of-variable formula.}}$

Example 1.1 Assume f is proper, i.e., $\forall K \subset V$ compact, $f^{-1}(K)$ is compact.

Then. $\int_U f^*(\phi \, dy) = \deg(f) \int_V \phi \, dy$.

Here: f is not assumed to be bijective.

$\deg(f)$ "=" the # of $f^{-1}(y)$ for a "generic" $y \in V$.

$\boxed{\text{degree theory.}}$

Example 2. $U \subseteq \mathbb{R}^2$ open.

$\Omega^0(U) :=$ vector space of $\mathbb{R}$-valued smooth functions on U.

$\Omega^1(U) := f_1 dx_1 + f_2 dx_2$, $f_i \in \Omega^0(U)$ $\forall i = 1, 2$.

$\Omega^2(U) := f \, dx_1 \wedge dx_2$

Here: $dx_1, dx_2, dx_1 \wedge dx_2$ are called "differential forms".

Def^n $d: \Omega^0(U) \to \Omega^1(U)$ $\qquad$ $d: \Omega^1(U) \to \Omega^2(U)$

$f \mapsto \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$ $\qquad$ $f_1 dx_1 + f_2 dx_2 \mapsto \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$

"de Rham cohomology"

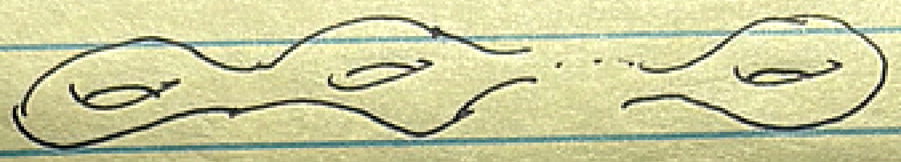
$H^i(U) := \ker(d: \Omega^i(U) \to \Omega^{i+1}(U)) / \mathrm{Im}(d: \Omega^{i-1}(U) \to \Omega^i(U))$.

Exercise: Check that $d^2 = 0$, here $\Omega^j(U) := \{0\}$ for $j \neq 0, 1, 2$.

Fact: This is a "topological invariant".

Goal Use de Rham theory to study topology of manifolds.

e.g. For a g-holed torus Σ_g,



$$\begin{cases} H^0(\Sigma_g) = \mathbb{R} \\ H^1(\Sigma_g) = \mathbb{R}^{2g} \\ H^2(\Sigma_g) = \mathbb{R}. \end{cases}$$

Course outline

- Part I. Multilinear algebra: exterior algebra of $\mathbb{R}^n$
- Part III: Integration of differential forms on ~~manifolds~~ $\mathbb{R}^n$
- Part II: Differential forms on ~~manifolds~~ Euclidean spaces
- Part IV: Differential forms on manifolds
- Part V: De Rham theories.

Prerequisite Multivariable calculus, linear algebra, basic topology and manifold theory.

Lecture 1 Linear algebra, tensors

• $\boxed{\text{Check list for basic concepts}}$

vector spaces; 0 vector; linear independence (If $\sum_{i=1}^{k} c_i v_i = 0$, then $c_i = 0$); span, basis; dimension (# of basis); subspace; linear map, its kernel & image, for $A: V \to W$, $\dim(V) = \cancel{\ker}\ \dim\ker(A) + \dim \operatorname{im}(A)$;

matrix coefficients: $v_1, \dots, v_n$, $w_1, \dots, w_m$ basis,

$$A v_j = \sum a_{ij} w_i \Rightarrow A = (a_{ij})_{ij}.$$

inner product: $B: V \times V \to \mathbb{R}$ w/
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$
- $B(v_1, v_2) = B(v_2, v_1)$
- $B(v,v) \geq 0$, '$=0$' $\Leftrightarrow v = 0$.

Exercise: Refresh your memory about linear algebra. ← (A) Quotient spaces

Def^n. V vector space, $W \subseteq V$ subspace.

$\boxed{W\text{-coset}}$ is a set of the form $v + W := \{v + w \mid w \in W\}$

Property. ① If $v_1 + W = v_2 + W$, then $v_1 - v_2 \in W$, & the converse holds

② $V = \bigsqcup W$-cosets.

Def^n. Quotient V/W: as a set = the set of W-cosets.

addition: $(v_1+W)+(v_2+W) = (v_1+v_2)+W$

scaler multiplication: $\lambda\cdot(v+W) = \lambda v + W$

proof of well-definedness: If $v_1+W = v_1'+W \iff v_1-v_1' \in W$

$\Rightarrow (v_1+W)+(v_2+W) = (v_1'+W)+(v_2+W)$

$= (v_1+v_2)+W \quad = (v_1'+v_2)+W$, similarly for v_2;

~~similarly for v_2~~ For scaler multiplication, if $v_1-v_1' \in W$,

then $\lambda(v_1-v_1') = \lambda v_1 - \lambda v_1' \in W \Rightarrow \lambda v_1 + W = \lambda v_1' + W$. □

Def^n. Quotient map $\pi: V \to V/W$ $\Rightarrow$ check: it's a linear map.

$v \mapsto v+W$

Prop^n. $\ker(\pi) = W$.

proof. If $v \in \ker(\pi)$, then v lies in the ~~coset~~ W-coset of $0 \Rightarrow v-0 = v \in W$;

if $v \in W$, certainly $\pi(v) = 0$. □

Accordingly, $\dim(V) = \dim(V/W) + \dim(W)$ as π is surjective.

Prop^n. If $A: V \to U$ linear, $W \subseteq \ker(A)$, then A factors as

$V \xrightarrow{\pi} V/W \xrightarrow{\bar{A}} U$. $\Rightarrow$ exercise.

(B) Dual vector space

~~Def^n. $l: V \to \mathbb{R}$ is called a linear function if~~

~~$l(\lambda v_1 + \lambda v_2)$~~

Def^n. $V \Rightarrow$ vector space. $V^{\vee} = V^*$ $\Rightarrow$ dual, consists of

① as a set, all linear functions $l: V \to \mathbb{R}$.

② addition: $(l_1+l_2)(v) := l_1(v)+l_2(v)$;

③ scaler multiplication: $(\lambda l)(v) := \lambda\, l(v)$.

Exercise. Verify that $V^* = V^{\vee}$ is a vector space.

Let $e_1, \dots, e_n$ be a basis of V.

Def^n. $e_i^{\vee}(c_1e_1 + \dots + c_ne_n) = c_i$, $e_i^{\vee}: V \to \mathbb{R}$.

Prop^n. $e_1^{\vee}, \dots, e_n^{\vee}$ form a basis of $V^{\vee}$.

proof. ① Linear independence: If $\sum c_i e_i^{\vee} = 0$, feed in $e_j \Rightarrow c_j = 0$ for any $j = 1, \dots, n$

② Spanning property: given $l: V \to \mathbb{R}$,
consider $\sum_{j=1}^{n} l(e_j) e_j^\vee$.
Then for any $v = \sum_{k=1}^{n} c_k e_k$, $l(v) = \sum_{k=1}^{n} c_k l(e_k)$
$= \left(\sum_{j=1}^{n} l(e_j) e_j^\vee\right)\left(\sum_{k=1}^{n} c_k e_k\right)$. □

$\boxed{\text{transpose}}$

Def^n. $A: V \to W$, its $\boxed{\text{dual}}$ $A^\vee: W^\vee \to V^\vee$ is defined as
$(l: W \to \mathbb{R}) \mapsto (l \circ A: V \to \mathbb{R})$.

Exercise. Check that A is linear.

Prop^n. If $A = (a_{ij})$ in basis $\{e_1, \dots, e_n\}$, $\{f_1, \dots, f_m\}$,
then $A^\vee = (a_{ij})^t$ in basis $\{f_1^\vee, \dots, f_m^\vee\}$, $\{e_1^\vee, \dots, e_n^\vee\}$.

proof. $\cancel{(A^\vee(f_i^\vee))(e_j) = f_i^\vee(Ae_j)}$
$\cancel{= f_i^\vee(\sum a_{ij} f_j)}$
$(A^\vee f_i^\vee)(e_j) = f_i^\vee(Ae_j) = f_i^\vee\left(\sum_{i=1}^{m} a_{ji} f_i\right) = a_{ji}$
$\Rightarrow$ $A^\vee$ is expressed as (a_{ji}), the transpose. □

(C) Tensors.

Def^n. (k-tensor). $V^k := \underbrace{V \times \cdots \times V}_{k \text{ times}}$
A function $T: V^k \to \mathbb{R}$ is called linear in its ith variable if
$\forall v_1, \dots, \hat{v}_i, \dots, v_k$, $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is in $V^\vee$.
A k-tensor is $T: V^k \to \mathbb{R}$ which is linear in all entries. $\boxed{k\text{-linear}}$

Example. • Convention: 0-tensors are identified w/ $\mathbb{R}$
• 1-tensor $\iff V^\vee$; ~~2-tensors~~
$\mathcal{L}^k(V) \Rightarrow$ all k-tensors on V, forms a vector space.
$(T_1 + T_2)(v_1, \dots, v_k) = T_1(v_1, \dots, v_k) + T_2(v_1, \dots, v_k)$, $(\lambda T)(v_1, \dots, v_k) = \lambda \cdot T(v_1, \dots, v_k)$.

Def^n. multi-index of n of length k $\iff$ $(i_1, \dots, i_k)$ s.t. $\forall 1 \le j \le k$, $1 \le i_j \le n$.

Example. # of multi-index of n of length $k = n^2$, corresponding to (i, j) w/ $1 \le i, j \le n$.
$\Rightarrow$ # ··· of n of length $k = n^k$. □

Lemma. $T: V^k \to \mathbb{R}$ is uniquely determined by its value $T_I := T(e_{i_1}, \dots, e_{i_k})$.

where $e_1, \dots, e_n$ is a basis. I ranges over all multi-index of n of length k.

proof. Induction on k. For $k=0$ (also 1), follows from previous discussion.

Assuming true for all ~~tensors~~ m-tensors with $m \le k-1$.

For a k-tensor T, consider the n $(k-1)$-tensors

$$T(e_1, -, \dots, -),\ T(e_2, -, \dots, -),\ \dots,\ T(e_n, -, \dots, -).$$

~~Then $\forall\ 1 \le j \le n$, $T(e_j$,~~

Then for $v = c_1 e_1 + \dots + c_n e_n$, $T(v, v_1, \dots, v_{n-1}) \overset{\text{linearity}}{=} \sum_{i=1}^{n} c_i T(e_i, v_1, \dots, v_{n-1})$,

and apply the induction hypothesis. $\square$

Def^n. (Tensor product). Suppose T_1 is a k-tensor, T_2 is an l-tensor.

Then $T_1 \otimes T_2$ is the $(k+l)$-tensor defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = T_1(v_1, \dots, v_k)\, T_2(v_{k+1}, \dots, v_{k+l}).$$

Properties.

① Associativity: $(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3)$

② Distributivity of scalar multiplication: $\lambda(T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2)$

③ Left + Right distributive laws: $\begin{cases} T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3 \\ (T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3 \end{cases}$

Exercise. Prove ①–③.

Def^n. A k-tensor T is called decomposable iff $T = l_1 \otimes \dots \otimes l_k$ for $l_1, \dots, l_k \in V^\vee$.

Example. $e_I^* := e_{i_1}^* \otimes \dots \otimes e_{i_k}^*$, for a ~~multiind~~ multi-index $I = (i_1, \dots, i_k)$.

Then $e_I^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & \text{if } I = J = (j_1, \dots, j_k) \\ 0 & \text{otherwise.} \end{cases}$

Thm. The k-tensors e_I^* ~~forms of~~ are a basis of $\mathcal{L}^k(V)$.

proof. Linear independence. If $\sum c_I e_I^* = 0$, test it against $(e_{j_1}, \dots, e_{j_k})$,

we see that $c_I \equiv 0$.

Spanning property: Consider $T' := \sum_I T(e_{i_1}, \dots, e_{i_k}) e_I^*$ for a k-tensor.

Then for any k-tuple $(v_1, \dots, v_k)$, $T(v_1, \dots, v_k) = T'(v_1, \dots, v_k)$. $\square$

Def^n. (Pullback) Let $A: V \to W$ be a linear mapping.

Then $A^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$

is defined by $(A^*T)(v_1, \dots, v_k)$
$:= T(Av_1, \dots, Av_k)$, called the pullback induced by A.

Properties. ① A^* is a linear mapping.
② $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
③ For linear mappings $A: V \to W$, $B: U \to V$, (another vector space)
we have $(AB)^* T = B^* A^* T$.

Exercise. Prove ① – ③.

§2. Alternating tensors, wedge product

Def^n. (Permutation group)
$\Sigma_k := \{1, \dots, k\}$ set of k elements.
permutation $\Leftrightarrow$ $\sigma: \Sigma_k \to \Sigma_k$ a bijection
$S_k :=$ permutation group of Σ_k, consisting of σ's w/.
① composition: $(\sigma_1 \circ \sigma_2)(i) = \sigma_1(\sigma_2(i))$,
② inverse: $\sigma(i) = j \Leftrightarrow \sigma^{-1}(j) = i$.
} group law. Rmk. $|S_k| = k!$

Def^n. ((Elementary) transposition) For $1 \le i < j \le k$,

$$\tau_{ij}(l) = \begin{cases} j & \text{if } l = i \\ i & \text{if } l = j \\ l & \text{if } l \ne i, j \end{cases}$$

It is called elementary if $j = i+1$.

Lemma. Any $\sigma \in \Sigma_k$ can be written as a composition of (elementary) transpositions.

proof. Induction on the length $l :=$ # of elements moved by σ) of σ.
For $l(\sigma) = 0$, trivial; assuming true for $l(\sigma) = m-1$.
Given $\sigma \in \Sigma_k$ s.t. $l(\sigma) = m$, taking $i \in \Sigma_k$ with $\sigma(i) = j$,
we see that $l(\sigma \circ \tau_{ij}) = m-1$. Apply inductive hypothesis.
On the other hand, $\tau_{ij} = \tau_{j-1,j} \tau_{i,j-1} \tau_{j-1,j}$,
so $\boxed{\text{Exercise}}$ any τ_{ij} is a composition of elementary transpositions. □

Def^{n}. Given $\sigma \in S_k$, $\mathrm{sign}(\sigma) := (-1)^{\#\{(i,j) \mid 1 \le i<j\le k,\ \sigma(i)>\sigma(j)\}}$

Equivalently, $\cancel{(-1)^{\mathrm{sign}(\sigma)}}\ \mathrm{sign}(\sigma) := \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$. $\boxed{\text{ALSO: } \mathrm{sign}(\sigma) = (-1)^{\sigma}}$

Claim. $\sigma \mapsto \mathrm{sign}(\sigma)$ defines a group homomorphism $S_k \to \mathbb{Z}/2$.

proof.

$$\cancel{(-1)^{\mathrm{sign}(\tau\sigma)}}\ \mathrm{sign}(\tau\sigma) = \prod_{i<j} \frac{x_{\tau(\sigma(i))} - x_{\tau(\sigma(j))}}{x_i - x_j}$$

$$= \prod_{i<j} \frac{x_{\tau(\sigma(i))} - x_{\tau(\sigma(j))}}{x_{\sigma(i)} - x_{\sigma(j)}} \cdot \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

$$= \prod_{i<j} \frac{x_{\tau(\sigma(i))} - x_{\tau(\sigma(j))}}{x_{\sigma(i)} - x_{\sigma(j)}} \cdot \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

The first term $= \cancel{(-1)^{\mathrm{sign}}}\ \mathrm{sign}(\tau)$ by renumbering, so R.H.S. $= \cancel{\mathrm{sig}}\ \mathrm{sign}(\tau)\cdot\mathrm{sign}(\sigma)$ $\square$

Cor. If σ is the product of an odd # of transpositions, $\mathrm{sign}(\sigma) = -1$;

- - - - even - - - - - - , $\mathrm{sign}(\tau) = 1$. $\square$

Given $\sigma \in S_k$, it acts on k-tensors by

$$T \in \mathcal{L}^k(V) \mapsto T^{\sigma} \in \mathcal{L}^k(V), \text{ where}$$

$$T^{\sigma}(v_1, \cdots, v_k) = T(v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(k)}).$$

Prop^{n}.

(1) If $T = l_1 \otimes \cdots \otimes l_k$, then $T^{\sigma} = l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(k)}$

(2) $T \mapsto T^{\sigma}$ is a linear map $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$

(3) If $\sigma, \tau \in S_k$, then $T^{\sigma\tau} = (T^{\sigma})^{\tau}$.

proof.

(1) $\cancel{T^{\sigma}}\ l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(k)}(v_1, \cdots, v_k)$

$= l_{\sigma(1)}(v_1) \cdots l_{\sigma(k)}(v_k)$

(apply σ^{-1}, reindexing) $= l_1(v_{\sigma^{-1}(1)}) \cdots l_k(v_{\sigma^{-1}(k)}) = T^{\sigma}(v_1, \cdots v_k)$.

(2) Definition.

(3) $T^{\sigma\tau}(v_1, \cdots v_k) = T(v_{(\sigma\tau)^{-1}(1)}, \cdots, v_{(\sigma\tau)^{-1}(k)})$

$= T(v_{\tau^{-1}\sigma^{-1}(1)}, \cdots, v_{\tau^{-1}\sigma^{-1}(k)})$

$= \cancel{T^{\tau}v} = \left((v_1, \cdots, v_k) \mapsto T(v_{\sigma^{-1}(1)}, \cdots v_{\sigma^{-1}(k)})\right)^{\tau}$

$= (T^{\sigma})^{\tau}$. $\square$

Def^n. A k-tensor $T\in\mathcal{L}^k(V)$ is called alternating iff $T^\sigma=(-1)^\sigma T$ $\forall\sigma\in S_k$.

This set is denoted by $\mathcal{A}^k(V)$.

The following definition introduces a way to construct elements in $\mathcal{A}^k(V)$.

Rmk This is the same as asking $T^{\tau_{i,i+1}}=-T$ for any $1\le i\le n-1$.

Def^n. $\boxed{\text{Alternation operation}}$ on $\mathcal{L}^k(V)$ is defined to be

$$Alt(T):=\sum_{\tau\in S_k}(-1)^\tau T^\tau.$$

Prop^n. (1) ~~Alt~~ $Alt(T)$ is an alternating tensor.

(2) If $T\in\mathcal{A}^k(T)$, then $Alt(T)=k!\,T$.

(3) $Alt(T^\sigma)=Alt(T)^\sigma$

(4) Alt is linear.

proof. (1) $Alt(T)^\sigma=\sum_{\tau\in S_k}(-1)^\tau T^{\tau\sigma}$

$=(-1)^\sigma\sum_{\tau\in S_k}(-1)^{\tau\sigma}T^{\tau\sigma}=(-1)^\sigma Alt(T)$.

(2) Note that if $T\in\mathcal{A}^k(T)$, then $T^\tau=(-1)^\tau T$.

Because $|S_k|=k!$, we have $Alt(T)=k!\,T$.

(3) $Alt(T^\sigma)=\sum_{\tau\in S_k}(-1)^\tau T^{\sigma\tau}=\left(\sum_{\tau\in S_k}(-1)^\tau T^\tau\right)^\sigma=Alt(T)^\sigma$.

(4) Direct verification. □

Def^n. For a multi-index $I=(i_1,\dots,i_k)$, write

$e_I^*=e_{i_1}^*\otimes\cdots\otimes e_{i_k}^*$, for $e_1,\dots,e_n$ being a basis of V.

Define $\psi_I=Alt(e_I^*)$.

Prop^n (1) $\psi_{I^\sigma}=(-1)^\sigma\psi_I$

(2) If $\exists\, i_s=i_t$ for $s\ne t$, then $\psi_I=0$.

(3) For $I=(i_1,\dots,i_k)$, $J=(j_1,\dots,j_k)$ s.t. $i_1<\cdots<i_k$, $j_1<\cdots<j_k$, (called strictly increasing)

then $\psi_I(e_{j_1},\dots,e_{j_k})=\begin{cases}1, & I=J\\ 0, & I\ne J\end{cases}$.

proof. (1) Because $\psi_{I^\sigma}=(\psi_I)^\sigma$ and ψ_I is $Alt(e_I^*)$,

we see that $\psi_{I^\sigma}=(-1)^\sigma\psi_I$.

(2) If $i_s=i_t$, then $\psi_I=\psi_I^{\tau_{s,t}}$,

but $\psi_I^{\tau_{s,t}}=(-1)\psi_I$ $\Rightarrow\psi_I=0$.

(3) If $I = J$, there is only one term in $Alt(e_I^*)$ pairing with e_J to be nonzero, and is equal to 1.

If $I \neq J$, every term is zero. □

Prop^n. The alternating tensors ψ_I w. I strictly increasing is are a basis of $A^k(V)$.

proof. Given $T \in A^k(V) \subseteq \mathcal{L}^k(V)$, we can write it as

$$T = \sum a_I e_I^*.$$

Because $Alt(T) = k!T$ and Alt is linear, we have

$$T = \frac{1}{k!} Alt(T) = \frac{1}{k!} \sum_I a_I \psi_I. \quad \Rightarrow \text{spanning property}$$

On the other hand, (3) of the previous Prop^n. implies linear independence. □

Exercise. Show that $\dim A^k(T) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.