Topic 2: transversality

The discussion here is on the classical side, so we don't need to do any virtual stuff.

1. Transversality via perturbation

We follow the exposition in the work of Floer–Hofer–Salamon [FHS95]. The textbooks [MS04] and [AD14] also contain detailed treatment of this classical topic.

Let (M, ω) be a compact symplectic manifold and $H_t : S^1 \times M \to \mathbb{R}$ (non-degenerate), J are as before. Consider the space

$$\mathcal{H} := \{G_t \in C^\infty(S^1 \times M, \mathbb{R}) | G_t = H_t \text{ near all } x : S^1 \to M \text{ with } \dot{x}(t) = X_{H_t}(x(t))\}.$$

Fix $\dot{x}^{\pm}(t) = X_{H_t}(x^{\pm}(t))$. Consider the moduli space of Floer trajectories

$$\mathcal{M}(x^-, x^+, G_t, J)$$

which consists of

$$u : \mathbb{R} \times S^1 \to M, \qquad \partial_s u + J(u)(\partial_t u - X_{G_t}(u)) = 0$$

such that

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t).$$

Definition 1.1. *$\mathcal{M}(x^-, x^+, G_t, J)$ is called* regular *if for any $u \in \mathcal{M}(x^-, x^+, G_t, J)$, the linearized operator*

$$D_u : W^{1,p}(\mathbb{R} \times S^1, u^*TM) \to L^p(\mathbb{R} \times S^1, u^*TM)$$

is surjective.

Here $p > 2$, or we can work with $W^{k,2} \to W^{k-1,2}$ with $k \geq 2$. It is a consequence of the implicit function theorem that if $\mathcal{M}(x^-, x^+, G_t, J)$ is regular, it is a smooth manifold.

Remark 1.2. *$\mathcal{M}(x^-, x^+, G_t, J)$ defined as above may have multiple path-connected components with varying Fredholm indices. It can be exhibited as a countable union ranging over $\pi_2(M)$. This is mostly relevant when discussing compactness properties of moduli spaces, as these homotopy classes arise in the formulas of topological energy. We abuse the notations in this lecture because it is irrelevant for most of the arguments.*

Definition 1.3. *Given a Banach space X, a subset $U \subset X$ is called* residual *or* of second category *if it contains an intersection of countably many open and dense subsets.*

By Baire's category theorem, every residual set is dense. For such a U, elements in U are also called *generic*. The goal of this section is the following.

Theorem 1.4. *The subset of $\mathcal{H}$ consisting of elements G_t such that $\mathcal{M}(x^-, x^+, G_t, J)$ is regular is of second category.*

In other words, for a generic perturbation of H_t, we can ensure that the moduli space of Floer trajectories is a smooth manifold. We need some standard results before embarking the proof.

Theorem 1.5 (Carleman Similarity Principle)**.** *Consider the equation for $u : B_\epsilon \to \mathbb{C}^n$ over the ball of radius ϵ in the complex plane*

$$\partial_s u + J(z)\partial_t u + C(z)u = 0$$

where $z = s + it$, $J(z)^2 = -Id$, and we consider the regularity class $u, J(z) \in W^{1,p}$ while $C(z) \in L^p(B_\epsilon, GL(n, \mathbb{C}))$. Then for u with $u(0) = 0$, there exists $0 < \delta \leq \epsilon$ with a map $\Phi \in W^{1,p}(B_\delta, GL(n, \mathbb{C}))$ and a holomorphic map $v : B_\delta \to \mathbb{C}^n$ such that

$$u(z) = \Phi(z)v(z), \qquad v(0) = 0, \qquad \Phi(z)^{-1}J(z)\Phi(z) = i.$$

The Carleman Similarity Principle says that solutions to Cauchy–Riemann type equations are closely related to genuine holomorphic maps. In particular, for any u with $\partial_s u + J(z)\partial_t u + C(z)u = 0$ vanishing at a point in B_δ with infinite order, we necessarily have $u \equiv 0$. A more general form of this observation is the following.

Proposition 1.6 (Unique continuation)**.** *Let $J(z, x)$ be a family of almost complex structures on $\mathbb{C}^n$ parametrized by $(z, x) \in \mathbb{C} \times \mathbb{C}^n$ of regularity class $W^{1,p}$ and $C : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ is of class $W^{1,p}$. Then for any two $u_1, u_2 \in W^{1,p}$ solving*

$$\partial_s u + J(z, u)\partial_t u + C(z, u) = 0,$$

over some open subset of $\mathbb{C}$, the set on which $u_1 = u_2$ up to infinite order is open and closed.

The proof of this fact is left as an exercise.

Exercise 1.7. *Write down the linear Cauchy–Riemann type equation satisfied by the difference $u_1 - u_2$ and deduce the Unique Continuation statement from the Carleman Similarity Principle.*

Going in a different direction, the following is the analogue of Sard's theorem in infinite dimensions, which was proved by Smale.

Proposition 1.8 (Sard–Smale Theorem)**.** *Let X and Y be separable Banach spaces and let $U \subset X$ be an open subset. Suppose $F : U \to Y$ is a Fredholm map of class C^l such that $l \geq \max\{1, \mathrm{ind}(F) + 1\}$. Then the set of regular values of F in Y is residual.*

Here, $y \in Y$ is said to be a regular value of F if $y \in \mathrm{im}(F)$, and for any $x \in F^{-1}(y)$, the differential $d_x F$ is surjective. For a full proof, one can consult [MS04, Theorem A.5.1].

The key idea of the generic transversality result is to apply the Sard–Smale Theorem to the universal moduli space

$$\mathcal{M}(x^-, x^+, \mathcal{H}, J) := \{(u, G_t) \mid G_t \in \mathcal{H}, u \in \mathcal{M}(x^-, x^+, G_t, J)\}.$$

Note that it admits a map

$$\pi : \mathcal{M}(x^-, x^+, \mathcal{H}, J) \to \mathcal{H}, \qquad (u, G_t) \mapsto G_t.$$

The linearization of the map

$$C^\infty(\mathbb{R} \times S^1, M) \times \mathcal{H} \to \Gamma(\mathbb{R} \times S^1, u^*TM)$$
$$(u, G_t) \mapsto \partial_s u + J(u)(\partial_t u - X_{G_t}(u))$$

can be written as

$$\tilde{D}_{u,G_t} : \Omega^0(\mathbb{R} \times S^1, u^*TM) \oplus \mathcal{H} \ni (\xi, \eta_t) \mapsto D_u\xi - J(u)X_{\eta_t}(u) = D_u\xi + \nabla\eta_t \in \Omega^0(\mathbb{R} \times S^1, u^*TM),$$

where we use the Riemannian metric $g = \omega(-, J-)$. As a result, given $(u, G_t) \in \mathcal{M}(x^-, x^+, \mathcal{H}, J)$, for (ξ, η_t) to lie in $\ker(d\pi)$, we see that $\eta_t = 0$ and $D_u\xi = 0$. In other words, we can identify $\ker(D_u)$ with $\ker(d\pi)$ at $(u, G_t) \in \mathcal{M}(x^-, x^+, \mathcal{H}, J)$. Similarly, by working with the formal adjoint, we see that $\operatorname{coker}(D_u) \cong \operatorname{coker}(d\pi)$. We summarize the discussion as follows.

Lemma 1.9. *Let $\mathcal{H}_l$ be the Banach space*

$$\{G_t \in C^l(S^1 \times M, \mathbb{R}) | G_t = H_t \text{ up to order } l \text{ near all } x : S^1 \to M \text{ with } \dot{x}(t) = X_{H_t}(x(t))\}.$$

Consider the universal moduli space

$$\mathcal{M}(x^-, x^+, \mathcal{H}_l, J) := \{(u, G_t) \in W^{1,p}(\mathbb{R} \times S^1, M) \times \mathcal{H}_l \mid \overline{\partial}_{J,G_t} u = 0\}.$$

Then the map

$$\pi : \mathcal{M}(x^-, x^+, \mathcal{H}_l, J) \to \mathcal{H}_l$$
$$(u, G_t) \mapsto G_t$$

is Fredholm of class C^l, and the Fredholm index at (u, G_t), as a virtual vector space, can be identified with $\ker(D_u) - \operatorname{coker}(D_u)$. In particular, D_u is surjective at (u, G_t) iff $d\pi$ is surjective.

Proof. The regularity class of π follows from the regularity assumptions on G_t. The closedness of the image of the linearization of π is a consequence of elliptic regularity. The rest of the assertion has been discussed above. □

In view of this Lemma, as long as we can prove that $\mathcal{M}(x^-, x^+, \mathcal{H}_l, J)$ is a C^l Banach manifold, we can use the Sard–Smale Theorem to prove the generic regularity statement, with a caveat of going from C^l to C^∞, about which we will discuss later.

Lemma 1.10. *For any $(u, G_t) \in \mathcal{M}(x^-, x^+, \mathcal{H}_l, J)$, the linearization*

$$(\xi, \eta_t) \mapsto D_u\xi + \nabla\eta_t$$

is surjective.

Proof. Note that we only need to produce enough η_t such that $\nabla\eta_t$ can kill the obstruction space $\operatorname{coker}(D_u)$. In other words, we need to show that for any $\zeta \in C^l(\mathbb{R} \times S^1, u^*TM)$ such that

$$D_u^*\zeta = 0, \qquad \int_{\mathbb{R} \times S^1} d\eta_t(u)\zeta ds dt = 0, \quad \forall \eta_t \in \mathcal{H}_l,$$

we must have $\zeta \equiv 0$.

Note that any $\zeta \in L^q(\mathbb{R}\times S^1, u^*TM)$ with $D_u^*\zeta = 0$ must be smooth by elliptic regularity. Here q is the Hölder dual $1/p + 1/q = 1$. We work with $l \geq 2$. We continue the proof with the following steps.

- For any $u \in \mathcal{M}(x^-, x^+, \mathcal{H}_l, J)$, the subset of $\mathbb{R} \times S^1$ on which $\partial_s u$ vanishes is discrete. This follows from differentiating $\overline{\partial}_{J,G_t} u = 0$ in the s-direction and apply the Carleman Similarity Principle Theorem 1.4 to $\partial_s u$. Here we use the fact that $x^+ \neq x^-$.
- For $u \in \mathcal{M}(x^-, x^+, \mathcal{H}_l, J)$, denote by $R(u)$ the set of points $(s,t) \in \mathbb{R} \times S^1$ such that for any $s' \neq s$, we have $u(s',t) \neq u(s,t)$ and $x^{\pm}(t) \neq u(s,t)$. We claim that $R(u)$ is open and dense. Then openness follows from observing that the complement is closed, which is a consequence of elliptic regularity in view of the convergence discussion. For the density property, please refer to [FHS95, Theorem 4.3]. Morally speaking, this is a consequence of the fact that if we ask $u(s,t) = u(s',t)$ for $s < s'$, this necessities $u(s'',t) \equiv u(s,t)$ for all $s < s'' < s'$, which would imply that u is a constant.
- Now given a smooth ζ with $D_u^*\zeta = 0$, let's take a point $(s_0,t_0) \in R(u)$ such that $\partial_s u(s_0,t_0) \neq 0$. Then we can find a smooth function $\eta_t : S^1 \times M \to \mathbb{R}$ such that
  (1) for $t_0 \in S^1$, the differential $d\eta_t$ at $u(s_0,t_0)$ is equal to $g(-, \zeta(s_0,t_0))$;
  (2) η_t has support in $(t_0 - \epsilon, t_0 + \epsilon) \times \mathrm{im}\big(u|_{(s_0-\epsilon, s_0+\epsilon)\times S^1}\big)$;
  (3) $d\eta_t(u(s,t))\zeta(s,t) \geq 0$ holds everywhere.

  Indeed, because $(s_0,t_0) \in R(u)$, we can choose a cut-off function to extend $\zeta(s_0,t_0)$. Such cut-off functions can be used to approximate the δ-distribution multiplied by $\zeta(s_0,t_0)$ at $(t_0, u(s_0,t_0)) \in S^1 \times M$. Accordingly, the condition $\int_{\mathbb{R}\times S^1} d\eta_t(u)\zeta dsdt = 0$ implies that ζ vanishes up to infinite order at (s_0,t_0).
- Finally, note that the Carleman similarity principle can be applied again to the equation $D_u^*\zeta = 0$, so that we know that any ζ which vanishes up to infinite order at a point is actually 0. So we have shown the surjectivity of the linearized operator at (u, G_t).

□

Upgrading to the C^∞ world. This is known as the Taubes' trick. We can define

$$\mathcal{H}^{reg,K} := \{G_t \in \mathcal{H} \mid \tilde{D}_{u,G_t} \text{ is surjective for } u \in \mathcal{M}(x^-, x^+, G_t, J), |\partial_s u| \leq K\},$$

and consider the space

$$\mathcal{H}^{reg} := \bigcap_{K\in\mathbb{N}} \mathcal{H}^{reg,K}.$$

We want to show that for each $K \in \mathbb{N}$, the space $\mathcal{H}^{reg,K}$ is open and dense in $\mathcal{H}$.

For the openness of $\mathcal{H}^{reg,K} \subset \mathcal{H}$, it suffices to argue that the complement is closed. Indeed, if we have $\{(u_\nu, G_t^\nu)\}_\nu$ such that D_{u_ν,G_t^ν} admits $\eta_\nu \neq 0$ with $D^*_{u_\nu,G_t^\nu}\eta_\nu = 0$ with $(u_\nu, G_t^\nu) \xrightarrow{\nu\to\infty} (u, G_t)$, after normalizing η_ν to have norm 1 and passing to a subsquence, we can find $\eta_\nu \to \eta$ with $D^*_{u,G}\eta = 0$. So we see that $\mathcal{H}^{reg,K} \subset \mathcal{H}$ is open.

For density, note that we have

$$\mathcal{H}^{reg,K} = \big(\bigcap_l \mathcal{H}_l^{reg,K}\big) \cap \mathcal{H}.$$

By Lemma 1.10 and the Sard–Smale theorem, we know that $\mathcal{H}_l^{reg,K}$ is dense in $\mathcal{H}_l$. Given $G_t \in \mathcal{H}$, viewing it as an element in $\mathcal{H}_l$, we can choose a sequence $\{G_t^{\nu,l}\} \subset \mathcal{H}_l^{reg,K}$ converging in C^l. Then a diagonal argument applied to the family with indices ν and l produces a sequence, and we can approximate each of them by a smooth element in $\mathcal{H}_l^{reg,K}$ which is $2^{-\nu}$-close in the C^l-norm, which can be done due to the openness of regularity. Therefore, $\mathcal{H}^{reg,K} \subset \mathcal{H}$ is dense. □

Inspecting the proof, one notes that the condition $|\partial_s u| \leq K$ is not used: this means that the above proof is not correct on the nose. Where does it go wrong? The point is, we cannot choose the sequence $(u_\nu, G_t^\nu) \xrightarrow{\nu\to\infty} (u, G_t)$ directly in the proof of the openness. Instead, we can only find $G_t^\nu \xrightarrow{\nu\to\infty} G_t$ such that each G_t^ν admits $u_\nu \in \mathcal{M}(x^-, x^+, G_t^\nu, J)$ with D_{u_ν} acquires a nontrivial cokernel element. We need something more to ensure that after passing to a subsequence if necessary, we can extract an limit $u_\nu \xrightarrow{\nu\to\infty} u$. This is where the assumption $|\partial_s u| \leq K$ becomes necessary, and we will study it in more detail when discussing the compactness of moduli spaces.

2. Variants of perturbations

There are two notable variants of the transversality argument exhibited above. It seems best to leave them as exercises.

In the first setting, we consider the *continuation map* equation. For that, let's choose a pair of non-degenerate Hamitonians $H_t^\pm$ and a $\mathbb{R}$-family of Hamiltonians $H_{s,t}$ such that

$$\lim_{s\to\pm\infty} H_{s,t} = H_t^\pm.$$

Then the continuation equation is defined to be

$$u : \mathbb{R} \times S^1 \to M, \qquad \partial_s u + J(u)(\partial_t u - X_{H_{s,t}}(u)) = 0,$$

where $X_{H_{s,t}}$ is defined by

$$dH_{s,t} = \omega(-, X_{H_{s,t}}).$$

Of course, we need to impose asymptotic conditions

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t)$$

where now $x^\pm(t)$ are respectively closed 1-periodic orbits of $X_{H_t^\pm}$. We denote the moduli space of such solutions by

$$\mathcal{M}(x^-, x^+, H_{s,t}, J).$$

Then we can consider the space

$$\mathcal{H}(H^\pm) := \{H_{s,t} : \mathbb{R} \times S^1 \times M \to \mathbb{R} \mid \lim_{s\to\pm\infty} H_{s,t} = H_t^\pm\}.$$

And there is the universal moduli space

$$\mathcal{M}(x^-, x^+, \mathcal{H}(H^\pm), J).$$

consisting of pairs $(u, H_{s,t})$ solving the continuation equation.

Exercise 2.1. *Derive the linearization of the equation which defines the space* $\mathcal{M}(x^-, x^+, \mathcal{H}(H^\pm), J)$

$$(\xi, \eta_{s,t}) \mapsto D_u\xi + \nabla\eta_{s,t}$$

and find the correct domain of $\eta_{s,t}$.

Exercise 2.2. *Show that there exists a residual subset of* $\mathcal{H}(H^\pm)$ *such that for any* $H_{s,t}$ *therein,* $\mathcal{M}(x^-, x^+, H_{s,t}, J)$ *is regular.*

In fact, the proof of this exercise is easier than the case of Floer trajectories. The point is to choose cut-off functions in both the s and t directions, which is an additional flexibility as we allow $H_{s,t}$ to depend on s, to create an approximation of the delta functional .

In the second setting, we look at the so-called *somewhere injective* curves. Given a closed Riemann surface (Σ, j) and an almost complex manifold (M, J), a J-holomorphic map

$$u : \Sigma \to M, \qquad du \circ j = J \circ du$$

is called somewhere injective if there exists $z_0 \in \Sigma$ such that

- $du : T_{z_0}\Sigma \to T_{u(z_0)}M$ is injective;
- $u^{-1}(u(z_0)) = \{z_0\}$.

Now we work with a compact symplectic manifold (X, ω). Denote by $\mathcal{J}^l$ the space of ω-compatible almost complex structures of regularity class C^l. Then the statement we want is the following.

Proposition 2.3. *Given* $J \in \mathcal{J}$*, and* $u : (\Sigma, j) \to M$ *a somewhere-injective* J*-holomorphic map, the universal moduli space*

$$\mathcal{M}(\Sigma, \mathcal{J}^l, M) := \{(v, J') \in W^{1,p}(\Sigma, M) \times \mathcal{J}^l \mid dv \circ j = J' \circ du\}$$

is a smooth Banach manifold near (u, J).

Exercise 2.4. *Write down the linearization of the map*

$$W^{1,p}(\Sigma, M) \times \mathcal{J}^l \to \bigcup_{u \in W^{1,p}(\Sigma,M)} L^p(\Omega^{0,1}_\Sigma(\Sigma, u^*TM))$$

$$(u, J) \mapsto \frac{1}{2}(du + J \circ du \circ j).$$

Here $\cup_{u \in W^{1,p}(\Sigma,M)} L^p(\Sigma, u^*TM)$ *is the Banach vector bundle over* $W^{1,p}(\Sigma, M)$ *whose fiber over* u *is* $L^p(\Omega^{0,1}_\Sigma(\Sigma, u^*TM))$.

Schematically, the answer is given by

$$(\xi, Y) \mapsto D_u\xi + Y \circ du \circ j,$$

where $\xi \in W^{1,p}(\Sigma, u^*TM)$, D_u is the analogue of the linear Floer operator with $H_t \equiv 0$, and Y is an infinitesimal almost complex structure compatible with ω:

$$Y \in \mathrm{End}(TM), \qquad J \circ Y + Y \circ J = 0, \qquad \omega(-, Y-) + \omega(Y-, -) = 0.$$

Exercise 2.5. *Show that for the injective point z_0, if $\eta \neq 0$ satisfies $D_u^* \eta = 0$, then we can find Y as above which is supported near $u(z_0)$ such that*

$$\int_\Sigma \langle \eta, Y \circ du \circ j \rangle > 0.$$

This exercise is done by a very similar argument as in the proof of Lemma 1.10. The solution will lead to the proof of Proposition 2.3. As a consequence, by applying the Taubes' trick with the necessary ingredients about compactness of moduli spaces, one can show that the moduli space of *somewhere injective* J-holomorphic maps with J generic is regular.

Exercise 2.6. *A J-holomorphic map $u : (\Sigma, j) \to M$ is call multiply-covered if u can be written as a composition $u = v \circ \phi$ where $\phi : (\Sigma, j) \to (\Sigma', j')$ is a holomorphic branched cover of degree ≥ 2 and $v : (\Sigma', j') \to M$ is J-holomorphic. Understand why if u is multiply-covered, the construction in Exercise 2.5 fails to hold.*

3. The general principle

Given a Banach space X and a Banach vector bundle $E \to X$, elliptic moduli problems produce Fredholm sections

$$\begin{array}{c} E \\ \downarrow \uparrow s \\ X \end{array}$$

which come from elliptic partial differential equations. Usually, the Fredholm section s depends on certain data (Riemannian metric, almost complex structure, or simply some inhomogeneous terms), which can be parametrized by a Banach space P. Then the moduli problem admits an extension

$$\begin{array}{c} E_P \\ \downarrow \uparrow s_P \\ X \times P \end{array}$$

such that the universal moduli space

$$\mathcal{M}_P := s_P^{-1}(0)$$

inherits the projection map

$$\pi_P : \mathcal{M}_P \to P.$$

It can be checked directly that for $(x, p) \in \mathcal{M}_P$, the kernel and cokernel of

$$ds_P(x, p)|_{T_x X} : T_x X \to E_P|_{(x,p)}$$

exactly coincides with the kernel and cokernel of

$$d\pi_P : T_{(x,p)} \mathcal{M}_P \to T_p P,$$

provided that $\mathcal{M}_P$ is a smooth manifold. The perturbation space P usually introduces 0-th order perturbations of the differential operator, therefore, elliptic regularity ensures that $d\pi_P$ is Fredholm. As a result, generic transversality is usually proven by the following two steps.

(1) Show that ds_P is surjective. This can be summarized as *having enough perturbations*.
(2) Apply the Sard–Smale theorem and the identification of Fredholm indices to conclude. This is where the notion *generic* comes up.

Exercise 3.1. *Suppose X is a compact finite-dimensional manifold and let $E \to X$ be a smooth vector bundle. Show that the subset of $C^\infty(\Gamma(X,E))$ consisting of $s : X \to E$ transverse to the 0-section is residual, using $C^\infty(\Gamma(X,E))$ itself as the perturbation space P.*

Exercise 3.2. *Let M be a smooth compact manifold and suppose $f : M \to \mathbb{R}$ is a Morse function. Given two critical points $x^\pm$, we can consider*

$$\mathcal{M}(x^-, x^+, f) := \{(v, g) \mid v : \mathbb{R} \to M, g \text{ Riemannian metric }, \dot{v} + \nabla_g f(v) = 0, \lim_{s \to \pm\infty} = x^\pm\}$$

Show that $\mathcal{M}(x^-, x^+, f)_k$ which consists of $v \in W^{1,2}$ and $g \in \mathcal{G}_k :=$ space of C^k-metrics is a smooth Banach manifold, with

$$\pi_{\mathcal{G}_k} : \mathcal{M}(x^-, x^+, f)_k \to \mathcal{G}_k$$

Fredholm of index $\operatorname{ind}(x^+) - \operatorname{ind}(x^-)$. Then show that

$$\mathcal{M}(x^-, x^+, f, g) := \{v : \mathbb{R} \to M \text{ smooth} \mid \dot{v} + \nabla_g f(v) = 0, \lim_{s \to \pm\infty} v(s) = x^\pm\}$$

is a smooth manifold for generic (= residual subset) g.

In the above exercise, one can also perturb the negative gradient flow line equation using smooth functions which are supported away from the critical points of f, like what we did for Floer trajectories.

References

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