Topic 1: Index theory

Useful references:

(1) Salamon's notes [Sal99] provides a condensed overview of Hamiltonian Floer theory with some technical details. For a more detailed exposition, one can consult Audin–Damian's book [AD14].
(2) A beautiful exposition of Fredholm theory of elliptic operators over manifolds with cylindrical ends is provided in Donaldson's book [Don02, Chapter 3].
(3) A good place to look at general theory of elliptic operators on compact manifolds is Wells' book on complex manifolds [Wel08, Chapter IV].

1. Linear Floer equation

(1a) Linearizing the Floer equation.

Given $H_t : S^1 \times M \to \mathbb{R}$ and an ω-compatible almost complex structure J, recall that the Floer equation is of the form

$$u : \mathbb{R} \times S^1 \to M, \qquad \partial_s u + J(u)(\partial_t u - X_{H_t}(u)) = 0.$$

Write the right hand side of the equation more compactly as

$$\overline{\partial}_{H_t,J} : C^\infty(\mathbb{R} \times S^1, M) \to \Gamma(\mathbb{R} \times S^1, u^*TM).$$

Let's write down the linearization of $\overline{\partial}_{H_t,J_t}$. Choose a Riemannian metric $g = \omega(-, J-)$ on M, which defines the Levi-Civita connection ∇ and the exponential map. Given a section $\xi \in \Gamma(\mathbb{R} \times S^1, u^*TM)$, let's look at

$$\mathcal{F}_u(\xi) := \Phi_{\exp_u(\xi) \to u} \overline{\partial}_{H_t,J}(\exp_u(\xi)),$$

where $\Phi_{\exp_u(\xi) \to u}$ denotes the parallel transport from $\exp_u(\xi)$ to u along the shortest geodesic. Then the linearization of $\overline{\partial}_{H_t,J}$ at u along the direction of ξ is

$$D_u \xi := \frac{d}{ds}|_{s=0} \mathcal{F}_u(s\xi) = (\nabla_s + J(u)\nabla_t)\xi + \nabla_\xi(J)(\partial_t u - X_{H_t}(u)) - J(u)\nabla_\xi X_{H_t}(u).$$

Exercise 1.1. *Show that there exists a isomorphism of vector bundles*

$$\Psi(s,t) : \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to u^*TM,$$

such that $\Psi^*\omega = \omega_0$ and $\Psi^*J = J_0$, the standard symplectic and complex structures on $\mathbb{R}^{2n}$.

Using $\Psi(s,t)$, the pullback $\Psi^* D_u := \Psi^{-1} \circ D_u \circ \Psi$ of D_u is given by

$$\begin{aligned} \eta \mapsto (\partial_s + J_0\partial_t)\eta + \Psi^{-1}\big((\nabla_s + J(u)\nabla_t)\Psi + \nabla_{\Psi(-)}(J)(\partial_t u - X_{H_t}(u)) - J(u)\nabla_{\Psi(-)} X_{H_t}(u)\big)\eta \\ := (\partial_s + J_0\partial_t)\eta + S\eta, \end{aligned}$$

for $\eta \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. If for $s \gg 1$ we have $\Psi(s,t) = \Psi^+(t)$, and for $s \ll -1$ we have $\Psi(s,t) = \Psi^-(t)$, we see that

$$S^\pm(t) := \lim_{s \to \pm\infty} S(s,t) = \Psi^{-1} J_\pm(t)(\nabla_t \Psi - \nabla_{\Psi(-)} X_{H_t}(u)) = J_0(\Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t}(u)),$$

where we use the fact that $\lim_{s\to\pm\infty} \partial_t u(s,t) = X_{H_t}$, corresponding to $\lim_{s\to\pm\infty} u(s,t)$ is given by 1-periodic orbits of $X_{H_t}(u)$. As a result, we see that at $s = \pm\infty$, $\Psi^* D_u$ can be written as

$$\eta \mapsto (\partial_s + J_0 \partial_t)\eta + S^\pm \eta = \partial_s \eta + (J_0 \partial_t + S^\pm)\eta.$$

Let's inspect the formula

$$\partial_t + \Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t}.$$

Given a loop $x : S^1 \to M$, we can define an element in $\Omega^1(S^1, \hom(x^*TM))$

$$-x^*(\nabla X_{H_t}).$$

Then we can see that $\partial_t + \Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t}$ is the formula for the connection $d - x^*(\nabla X_{H_t})$ conjugated by Ψ if there were a preferred trivialization of $x^*TM \cong S^1 \times \mathbb{R}^{2n}$ such that $d = \partial_t \otimes dt$.

(1b) Asymptotic operators and separation of variables.

The above calculation of the linearized Floer operator motivates the following definition.

Definition 1.2. *Let $x : S^1 \to M$ satisfy $\dot{x}(t) = X_{H_t}(x)$. The* asymptotic operator *of x with respect to a trivialization $\Psi : S^1 \times \mathbb{R}^{2n} \cong x^*TM$ is*

$$\begin{aligned} L_x : \Gamma(S^1, S^1 \times \mathbb{R}^{2n}) &\to \Gamma(S^1, S^1 \times \mathbb{R}^{2n}) \\ \eta &\mapsto J_0\left(\frac{d}{dt} + \Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t}\right). \end{aligned}$$

Lemma 1.3. *L_x is a self-adjoint opertor with respect to the inner product*

$$\langle \eta_1, \eta_2 \rangle = \int_{S^1} g_J(\eta_1, \eta_2) dt.$$

If x is a non-degenerate fixed point of ϕ_{H_t}, then L_x does not admit 0 as an eigenvalue.

Proof. Note that

$$\int_{S^1} g_J(\eta_1, J_0 \frac{d}{dt}\eta_2) dt = \int_{S^1} -g_J(J_0 \eta_1, \frac{d}{dt}\eta_2) dt,$$

and

$$0 = \int_{S^1} \frac{d}{dt} g_J(J_0\eta_1, \frac{d}{dt}\eta_2) = \int_{S^1} g_J(J_0 \frac{d}{dt}\eta_1, \frac{d}{dt}\eta_2) dt + g_J(J_0\eta_1, \frac{d}{dt}\eta_2) dt,$$

we see that

$$\int_{S^1} g_J(\eta_1, J_0 \frac{d}{dt}\eta_2) dt = \int_{S^1} g_J(J_0 \frac{d}{dt}\eta_1, \eta_2) dt.$$

On the other hand, $J_0(\Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t})$ is a symmetric matrix because Ψ is unitary and $\Psi^{-1} \nabla_{\Psi(-)} J_0 X_{H_t}$ comes from a Hessian. Therefore, L_x is self-adjoint.

Consider a family of symplectic matrices

$$\tilde{\Psi}(t) : [0,1] \to \mathrm{Sp}(2n)$$
$$t \mapsto \Psi \circ d\phi_H^t \circ \Psi^{-1},$$

which solves the equation

$$\frac{d}{dt}\tilde{\Psi} = -(\Psi^{-1} \circ \nabla_t \Psi - \Psi^{-1} \nabla_{\Psi(-)} X_{H_t})\tilde{\Psi}.$$

Choosing an identification $S^1 \cong [0,1]/\sim$, after lifting L_x to $[0,1]$, we can conjugate L_x using $\tilde{\Psi}$ to see that $\tilde{\Psi}^{-1} \circ L_x \circ \tilde{\Psi} = J_0 \frac{d}{dt}$. Then we see that $\eta : [0,1] \to \mathbb{R}^{2n}$ descends to an eigenfuntion of L_x iff $\eta(0) = \eta(1)$ and $\tilde{\Psi}^{-1}(\eta)$ is a nonzero constant vector in $\mathbb{R}^{2n}$. This means that $\eta(0) = \eta(1) = d\phi_{H_t}(\eta(0))$, so 1 is an eigenvalue of $d\phi_{H_t}$. This proves the second claim. □

From now on, we assume that L_x does not have 0 as an eigenvalue. Accordingly, there exists $\delta > 0$ such that if λ is an eigenvalue of L_x, we necessarily have $|\lambda| \geq \delta$. Now let's consider the cylinder with the operator

$$\frac{\partial}{\partial s} + L_x : \Gamma(\mathbb{R} \times S^1, \mathbb{R} \times S^1 \times \mathbb{R}^{2n}) \to \Gamma(\mathbb{R} \times S^1, \mathbb{R} \times S^1 \times \mathbb{R}^{2n}). \tag{1.1}$$

Lemma 1.4. *Suppose $\rho \in \Gamma(\mathbb{R} \times S^1, \mathbb{R} \times S^1 \times \mathbb{R}^{2n})$ is a compactly supported. Then there exists $f \in \Gamma(\mathbb{R} \times S^1, \mathbb{R} \times S^1 \times \mathbb{R}^{2n})$ such that $(\frac{\partial}{\partial s} + L_x)(f) = \rho$ and $\|f\|_{L^2} \leq \delta^{-1}\|\rho\|_{L^2}$.*

Proof. Denote by $\{\phi_\lambda\}$ a complete set of eigenvectors of L_x. Write

$$\rho = \sum_\lambda \rho_\lambda(s)\phi_\lambda, \qquad f = \sum_\lambda f_\lambda(s)\phi_\lambda$$

Using the eigen-function expansion. Then write

$$(\frac{\partial}{\partial s} + L_x)f = \sum_\lambda (f'_\lambda(s) + \lambda f_\lambda(s))\phi_\lambda,$$

so we would like to solve

$$f'_\lambda(s) + \lambda f_\lambda(s) = \rho_\lambda(s).$$

Then the solutions are of the form

$$f_\lambda(s) = e^{-\lambda s}\int_{-\infty}^{s} e^{\lambda\tau}\rho_\lambda(\tau)d\tau \text{ if } \lambda > 0,$$
$$f_\lambda(s) = -e^{-\lambda s}\int_{s}^{\infty} e^{\lambda\tau}\rho_\lambda(\tau)d\tau \text{ if } \lambda < 0.$$

Note that

$$(f'_\lambda(s))^2 + (\lambda f_\lambda(s))^2 = \rho_\lambda(s)^2 - 2\lambda f'_\lambda(s) f_\lambda(s) = \rho_\lambda(s)^2 - \frac{d}{ds}(f_\lambda(s)^2),$$

from which the integration gives us

$$\int (f'_\lambda(s))^2 + \int (\lambda f_\lambda(s))^2 = \int \rho_\lambda(s)^2.$$

Summing over λ we see that $\|f\|_{L^2} \leq \delta^{-1}\|\rho\|_{L^2}$. Although this only provides a distributional solution *a priori*, elliptic regularity ensures that f is actually smooth. □

Recall that $W^{1,2}(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n})$ is the Sobolev completion of $\Gamma(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n})$ under the norm $\|f\|_{L^2}+\|\nabla f\|_{L^2}$. We actually see that $\frac{\partial}{\partial s}+L_x$ extends to an operator

$$W^{1,2}(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n})\to L^2(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n}),$$

and the above lemma shows that it is surjective. The same proof shows that for any ρ, there is a *unique* f such that $(\frac{\partial}{\partial s}+L_x)(f)=\rho$. We record it as:

Proposition 1.5. $\frac{\partial}{\partial s}+L_x : W^{1,2}(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n})\to L^2(\mathbb{R}\times S^1,\mathbb{R}\times S^1\times\mathbb{R}^{2n})$ *is an isomorphism with an inverse* $Q: L^2\to W^{1,2}$ *such that* $\|Q\rho\|_{L^2}\le\delta^{-1}\|\rho\|_{L^2}$.

Exercise 1.6. *Consider the operator* $\frac{\partial}{\partial s}+L_x$ *restricted to the finite cylinder* $(-T,T)\times S^1$. *Show that if* $(\frac{\partial}{\partial s}+L_x)f=0$, *then*

$$\int_{(-T,T)\times S^1}|f|^2\le\frac{1}{1-e^{-2\delta}}\int_{(-T,-T+1)}|f|^2+\int_{(T,T-1)}|f|^2.$$

Also, all derivatives of f *decays exponentially in* T.

(1c) Linearized operators are Fredholm.

We wish to show that the pullback of the linearized Floer operator Ψ^*D_u, after being extended to an operator $W^{1,2}\to L^2$, is Fredholm. Actually, we would like to discuss a more general form of elliptic theory of Cauchy–Riemann operators.

Let (Σ,j) be a punctured Riemann surface such that the punctures are endowed with *cylindrical ends*, i.e., biholomorphic maps onto its image

$$\epsilon_i^+ : [0,\infty)\times S^1\to\Sigma \text{ or } \epsilon_j^- : (-\infty,0]\times S^1\to\Sigma$$

such that the limit is a puncture in Σ. Given a complex vector bundle $E\to\Sigma$ of real rank $2n$, a differential operator

$$D:\Omega^0(\Sigma,E)\to\Omega^{0,1}(\Sigma,E)$$

is called *of Cauchy–Riemann type* if for any smooth function $f\in C^\infty(\Sigma,\mathbb{C})$ and $s\in\Omega^0(\Sigma,E)$, we have

$$D(fs)=\overline{\partial}(f)s+fD(s), \tag{1.2}$$

where $\overline{\partial}=\frac{1}{2}(d+J\circ d\circ j)$ is the standard Cauchy–Riemann operator and J is the fiberwise complex structure on E. Going back to Σ, we consider its real-oriented blowups at the punctures, which can be thought of as the image of the circle S^1_j at ∞ of the cylindrical ends. We assume the following data:

1. for each puncture j, a trivialization of the vector bundle $\Phi_j^\pm : S^1\times\mathbb{R}^{2n}\cong E|_{S^1_j}$;
2. using the cylindrical coordinates and the trivialization $\Phi_j^\pm$, we require that the pullback of D can be written as
$$\partial_s+J_0(\partial_t+S_j(t)),$$
where $S_j(t)$ is a family of symmetric matrices such that $J_0(\partial_t+S_j(t))$ does not have 0 as an eigenvalue.

Of course, the linearized Floer operator is a special case.

Proposition 1.7. *Under the above assumptions, $D : W^{1,2} \to L^2$ is a Fredholm operator:*

- *the kernel* $\ker(D)$ *is finite-dimensional;*
- *the image of D is a closed subspace of L^2 of finite codimension in L^2, which agrees with the kernel of the formal adjoint D^*.*

Proof. For the first assertion, it suffices to show that the unit ball of $\ker(D)$ under the L^2-norm is compact. Suppose $\{f_k\}$ is a sequence such that $\|f_k\|_{L^2} \leq 1$ and $Df_k = 0$. As the Cauchy–Riemann operator is elliptic, we can find f_∞ such that $f_k \to f_\infty$ in L^2 over compact subsets of Σ. It suffices to check that the convergence holds over the whole Σ. This follows from the estimate in Exercise 1.6, which can be applied to the positive and negative case respectively.

For the second assertion, we construct an inverse of D modulo compact operators. Over the cylindrical ends, using Proposition 1.5, we can find $Q_j : L^2 \to W^{1,2}$ such that $(\partial_s + J_0(\partial_t + S_j(t))) \circ Q = Id$. In the complement of the cylindrical regions, we can choose a finite cover $\cup U_k$ and a smooth partition of unity subordinate to this cover $1 = \sum_k \beta_k$, such that the restriction of D along U_k admits a right inverse $\hat{Q}_k$, whose existence follows from the ellipticity of D. Extending the partition of unity by choosing $\beta_j : \Sigma \to [0,1]$ which is equal to 1 over image of ϵ_j, we consider

$$Q = \sum_j \beta_j Q_j + \sum_k \beta_k \hat{Q}_k.$$

Then one can check that

$$DQ = Id + \text{ compact operator.}$$

Then the second assertion follows from standard Fredholm theory and the fact that $\mathrm{coker}(D)$ can be identified with the L^2 complement of $\mathrm{im}(D)$. □

Exercise 1.8. *Recall the construction of paramatrices of elliptic operators.*

More generally, it can be shown that $D : W^{1,p} \to L^p$ for any $1 < p < \infty$ is a Fredholm operator, which uses Calderon–Zygmund estimates and deeper theory of Sobolev spaces. On the other hand, one can also work with $W^{k,2} \to W^{k-1,2}$ spaces with $k \geq 2$, for which the Fouriere analysis approach as above carries over without too much change except for keeping track of more derivatives. In any case, the point of working with $W^{1,p}$ with $p > 2$ or $W^{k,2}$ with $k \geq 2$ is to ensure the compact embedding $W^{1,p}, W^{k,2} \hookrightarrow C^0$.

2. Index calculation

The goal is to derive the Fredholm index of Cauchy–Riemann type operators discussed in the previous section.

(2a) Gluing and additivity of indices. Let (Σ_1, j_1) and (Σ_2, j_2) be punctured Riemann surfaces with cylindrical ends. Choosing a negative cylindrical end $\epsilon_1 : (-\infty, 0] \times S^1 \to \Sigma_1$ and a positive cylindrical end $\epsilon_2 : [0, +\infty) \times S^1 \to \Sigma_2$, consider

$$\Sigma_1^R := \Sigma_1 \setminus \epsilon_1((-\infty, -2R) \times S^1) \text{ and } \Sigma_2^R := \Sigma_2 \setminus \epsilon_2((2R, \infty) \times S^1).$$

Then we construct a new Riemann surface

$$\Sigma_1 \#_R \Sigma_2$$

where the gluing is performed under the isomorphism

$$[-2R, -R] \times S^1 \cong [R, 2R] \times S^1$$
$$(s, t) \mapsto (s + 3R, t)$$

over the cylindrical region. Now suppose (Σ_1, j_1) and (Σ_2, j_2) come with vector bundles $E_1 \to \Sigma_1$, $E_2 \to \Sigma_2$ and Cauchy–Riemann type operators D_1, D_2, together with the trivializations and asymptotic operators. If the data (E, Φ, S) agree over the cylindrical ends over which the gluing is performed, we obtain a CR type operator D_R over $\Sigma_1 \#_R \Sigma_2$.

Theorem 2.1. *For R sufficiently large, we have*

$$\text{ind}(D_R) = \text{ind}(D_1) + \text{ind}(D_2).$$

Proof. We first consider the case when both D_1 and D_2 are surjective. Define a smooth function β_1 on Σ_1 which is equal to 1 as $s \geq -R$ (extended by 1 across the complement of the cylindrical region) and has support within $s \geq -2R$, with the condition that $|\nabla \beta_1| \leq C/R$ for some constant $C > 0$. On the other hand, choose $\beta_2 : \Sigma_2 \to \mathbb{R}$ which is equal to $1 - \beta_1(s - 3R, t)$ over $[R, 2R] \times S^1$, and extended by 0 and 1 respectively over $[2R, \infty] \times S^1$ and the complement of the cylindrical region in Σ_2. Given $f_1 \in \Gamma(\Sigma_1, E_1)$ and $f_2 \in \Gamma(\Sigma_2, E_2)$, we can define a section over $\Sigma_1 \#_R \Sigma_2$ of the form

$$\beta_1 f_1 + \beta_2 f_2,$$

where we reparametrize $\beta_1 f_1$ over $[R, 2R] \times S^1$ using the shift $s \mapsto s + 3R$. We call this the *gluing* of two sections.

On the other hand, over $\Sigma_1 \#_R \Sigma_2$, we can choose two smooth functions as follows. For $\beta_1' : \Sigma_1 \#_R \Sigma_2 \to \mathbb{R}$, it is supported in $s \geq -R$ in the copy of Σ_1 and is equal to 1 in the complement of the cylindrical region; for $\beta_2' : \Sigma_1 \#_R \Sigma_2 \to \mathbb{R}$, it is supported in $s \leq R$ in the copy of Σ_2 and is equal to 1 in the complement of the cylindrical region. Then given a section f over $\beta_1' : \Sigma_1 \#_R \Sigma_2 \to \mathbb{R}$, we obtain a pair of sections

$$\beta_1' f \in \Gamma(\Sigma_1, E_1) \text{ and } \beta_2' f \in \Gamma(\Sigma_2, E_2),$$

which we call the *breaking* of a section.

Choosing bounded right inverses Q_1 and Q_2 of D_1 and D_2 respectively, consider the operator

$$\tilde{Q}_R := \text{gluing} \circ (Q_1 \oplus Q_2) \circ \text{breaking}.$$

Then $\tilde{Q}_R$ is an approximate right inverse to D_R in the following sense: there exists $C > 0$ such that

$$\|D_R \tilde{Q}_R - Id\|_{W^{1,2}} \leq \frac{C}{R}.$$

Then for R sufficiently large, we see that

$$Q_R := \tilde{Q} \sum_k (Id - D\tilde{Q})^k$$

is a bounded right inverse to D_R: the series $\sum_k (Id - D\tilde{Q})^k$ converges via the the comparison with geometric series, and recall that $\sum_k (1-x)^k = \frac{1}{1-(1-x)} = x^{-1}$. This shows that D_R is surjective.

We can define

$$\ker(D_1) \oplus \ker(D_2) \to \ker(D_R)$$
$$(f_1, f_2) \mapsto (Id - Q_R \circ D_R)(\beta_1 f_1 + \beta_2 f_2),$$

which is now shown to be an isomorphism for R sufficiently large. For injectivity, it follows from the unique continuation by restricting to the "bulk region" of Σ_1 and Σ_2 as the map sends the sections f_1 and f_2 to itself. For surjectivity, note that over the neck region $[0, 3R] \times S^1$, the glued operator D_R has the form

$$\partial_s + J_0(\partial_t + S(t)),$$

so we know that for any $f \in \ker(D_R)$, its restriction to the neck region satisfies the exponential decay in R. By looking at the projections to $\ker(D_1)$ and $\ker(D_2)$ of the breaking, we see that f can be arbitrarily close to an element from gluing by making R sufficiently large. Then a compactness argument by letting $R \to \infty$ shows that if there exists $f \in \ker(D)$ which does not come from gluing, this cannot happen. This establishes the surjectivity of gluing.

Finally, when D_1 and D_2 have nontrivial cokernels, we consider stabilization: choose finite dimensional vector spaces V_1 and V_2 with linear isomorphisms $\iota_1 : V_1 \cong \operatorname{coker}(D_1)$ and $\iota_2 : V_2 \cong \operatorname{coker}(D_2)$, and look at the operators

$$D_1 \oplus \iota_1, \qquad D_2 \oplus \iota_2.$$

Then the index of the glued operator is calculated as

$$\operatorname{ind}(D_R) = \operatorname{ind}(D_1 \oplus \iota_1) + \operatorname{ind}(D_2 \oplus \iota_2) - \dim V_1 - \dim V_2$$

by the additivity of Fredholm indices under direct sums. □

The above proof shows not only the numerical additivity of Fredholm indices. If we use the same notation $\operatorname{ind}(D)$ to denote the virtual vector space

$$\ker(D) - \operatorname{coker}(D),$$

the additivity relation

$$\operatorname{ind}(D_R) = \operatorname{ind}(D_1) + \operatorname{ind}(D_2).$$

remains to hold as virtual vector spaces.

Remark 2.2. *The same proof as in Theorem 2.1 shows the following:*

(1) *If $E_i \to \Sigma_i$ are equipped with group actions some by Lie group G, the Fredholm index $\operatorname{ind}(D_i)$ is a virtual G-representation, i.e., equivalence classes of difference of finite-dimensional G-representations. Then $\operatorname{ind}(D_R) = \operatorname{ind}(D_1) + \operatorname{ind}(D_2)$ holds in the representation ring of G.*

(2) *Given a* family *of Fredholm data (Σ_i, E_i, D_i) parametrized by compact topological spaces M_i, the Fredholm index defines a virtual vector bundle $\operatorname{ind}(D_i) \to M_i$. Then over the product $M_1 \times M_2$, we can perform the gluing of Riemann surfaces to obtain a family of Fredholm operators. Then $\operatorname{ind}(D_R) = \pi_1^* \operatorname{ind}(D_1) + \pi_2^* \operatorname{ind}(D_2)$ in the K-theory of $M_1 \times M_2$.*

Exercise 2.3. *Fill in the details of the second case.*

(2b) Conley–Zehnder indices. Consider the Riemann surface $\mathbb{C} \cong S^2 \setminus \{\infty\}$, for which ∞ is equipped with the positive cylindrical end $[0,\infty) \times S^1 \to \mathbb{C}$ taking (s,t) to $e^{s+2\pi i t}$. Under the cylindrical end, suppose that we are looking at the Cauchy–Riemann type operator on the trivial bundle $\underline{\mathbb{R}}^{2n}$

$$\partial_s + J_0(\partial_t + S(t)).$$

We extend it to an operator

$$\overline{\partial}_S : \Gamma(\mathbb{C}, \underline{\mathbb{R}}^{2n}) \to \Gamma(\mathbb{C}, \underline{\mathbb{R}}^{2n})$$

such that the restriction of $\overline{\partial}_S$ to the unit disc agrees with the standard Cauchy–Riemann operator $\partial_s + J_0\partial_t$. Then we know that $\overline{\partial}_S$ is a Fredholm operator after making the extension $W^{1,p} \to L^p$ or $W^{k,2} \to W^{k-1,2}$. On the other hand, by solving the equation with initial condition $\Psi(t) = Id$

$$\dot{\Psi}(t) = J_0 S(t)\Psi(t),$$

we obtain a 1-parameter family of symplectic matrices. The matrix $S(t)$ can then be recovered from $\Psi(t)$ by taking derivative in t.

For a path of symplectic matrices $\Psi(t) : [0,1] \to \mathrm{Sp}(2n)$ with $\Psi(0) = Id$ and $\Psi(1)$ does not have 1 as an eigenvalue, the *Conley–Zehnder index* μ_{CZ} of $\Psi(t)$ is characterized by the following properties:

(1) It's invariant under conjugations: for $\Phi(t) : [0,1] \to \mathrm{Sp}(2n)$. we have $\mu_{CZ}(\Phi\Psi\Phi^{-1}(t)) = \mu_{CZ}(\Psi(t))$.
(2) It's invariant under homotopy relative to end points.
(3) If $\Psi(t)$ does not have an eigenvalue on the unit circle for all t, then $\mu_{CZ}(\Psi(t)) = 0$.
(4) For $\Psi_1(t) : [0,1] \to \mathrm{Sp}(2n_1)$ and $\Psi_2(t) : [0,1] \to \mathrm{Sp}(2n_2)$, the direct sum $(\Psi_1 \oplus \Psi_2)(t) : [0,1] \to \mathrm{Sp}(2(n_1+n_2))$ satisfies $\mu_{CZ}(\Psi_1 \oplus \Psi_2) = \mu_{CZ}(\Psi_1) + \mu_{CZ}(\Psi_2)$.
(5) For a loop $\Phi : S^1 \to \mathrm{Sp}(2n)$ based at the identity, define $\mu(\Phi)$ to be the degree of the map $\det(\Phi)$. Then $\mu_{CZ}(\Phi\Psi) = 2\mu(\Phi) + \mu_{CZ}(\Psi)$.
(6) For a fixed symmetric matrix S, the path $t \mapsto \exp(tJ_0S)$ satisfies $\mu_{CZ}(\exp(tJ_0S)) = \frac{1}{2}\mathrm{sign}(S)$ if $\|S\| < 2\pi$ and sign denotes the signature.
(7) $(-1)^{\mu_{CZ}(\Psi(t))} = \mathrm{sign}\det(Id - \Psi(1))$.
(8) For the inverse path $\Psi^{-1}(t)$, we have $\mu_{CZ}(\Psi^{-1}(t)) = -\mu_{CZ}(\Psi(t))$.

Exercise 2.4. *Show that the above properties uniquely determine* μ_{CZ}.

Proposition 2.5. *Define $S(t)$ via $\dot{\Psi}(t) = J_0S(t)\Psi(t)$, then the Fredholm index of $\overline{\partial}_S$ satisfies the above properties is equal to $n - \mu_{CZ}(\Psi(t))$.*

Proof. (1), (2), (4) follows from standard properties of Fredholm indices. (3), (6), (7) follows from model calculations after performing suitable homotopies. For (5) and (8), they follow from the gluing formula of Fredholm indices of Cauchy–Riemann type operators. For a detailed proof of (5), see [Abo15, Proposition 1.4.10]. □

(2c) The index formula. Let (Σ, j) be a punctured Riemann surface with positive and negative cylindrical ends $\epsilon_i^+, \epsilon_j^-$. Suppose $E \to \Sigma$ is a Hermitian vector bundle of rank n. If $D : \Omega^0(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$ is a Cauchy Riemann type operator, and under the trivializations of E restricted to the cylindrical regions

$$\Phi_i^+ : [0,\infty) \times S^1 \times \mathbb{C}^n \xrightarrow{\sim} E|_{\epsilon_i^+([0,\infty)\times S^1)}$$
$$\Phi_j^- : (-\infty,0] \times S^1 \times \mathbb{C}^n \xrightarrow{\sim} E|_{\epsilon_j^-((-\infty,0]\times S^1)},$$

the operators are pulled back to

$$\overline{\partial}_{S_i^+} = \partial_s + J_0(\partial_t + S_i^+(t))$$
$$\overline{\partial}_{S_j^-} = \partial_s + J_0(\partial_t + S_j^-(t)),$$

we can produce the 1-parameter families of symplectic matrices $\Psi_i^+(t)$ and $\Psi_j^-(t)$.

Proposition 2.6. *The Fredholm index of D is*

$$\operatorname{ind}(D) = n\chi(\Sigma) + \sum_i \mu_{CZ}(\Psi_i^+) - \sum_j \mu_{CZ}(\Psi_j^-).$$

Proof. Using the index gluing formula, we see that

$$\operatorname{ind}(D) + \sum_i (n - \mu_{CZ}(\Psi_i^+)) - \sum_j (n + \mu_{CZ}(\Psi_j^-))$$

is equal to the Fredholm index of the standard $\overline{\partial}$-operator on the trivial bundle $\mathbb{C}^n$ over the compactification $\overline{\Sigma}$, which is $n(2 - 2g(\overline{\Sigma}))$. □

In particular, we see that the Fredholm index of the linearized Floer operator is

$$\mu_{CZ}(\Psi^+) - \mu_{CZ}(\Psi^-).$$

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