Topic 4: gluing

We continue the previous setup: (M, ω) is a compact symplectic manifold, with the simplifying assumption that $\omega|_{\pi_2(M)} = 0$, and $H_t : S^1 \times M \to \mathbb{R}$ is a 1-periodic nondegenerate Hamiltonian. We choose an ω-compatible almost complex structure J and assume that the moduli spaces

$$\mathcal{M}(x^-, x^+) := \mathcal{M}(x^-, x^+, H_t, J)$$

are all regular.

1. Generalities

Although $\mathcal{M}(x^-, x^+)$ may not be compact due to the existence of broken Floer trajectories as limits, the space

$$\overline{\mathcal{M}}(x^-, x^+) = \bigcup_{\substack{r \geq 0 \\ y_1, \dots, y_r}} \mathcal{M}(x^-, y_1) \times \mathcal{M}(y_1, y_2) \times \cdots \times \mathcal{M}(y_r, x^+)$$

is compact. The gluing construction gives rise to continuous injections

$$\mathcal{M}(x^-, y_1) \times \mathcal{M}(y_1, y_2) \times \cdots \times \mathcal{M}(y_r, x^+) \times [T_0, \infty]^r \to \overline{\mathcal{M}}(x^-, x^+)$$

which completely describe a neighborhood of $\overline{\mathcal{M}}(x^-, x^+)$ near its boundaries and corners. We focus on explaining how this can be done when $r = 1$, and we do not touch upon the issue of constructing smooth gluing maps including the points at ∞.

2. Gluing in an example

Suppose that to compactify $\mathcal{M}(x_1, x_3)$, we need to add in the broken configuration

$$\mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3).$$

Our goal is to describe a *gluing map* for $T_0 \gg 1$

$$\mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3) \times [T_0, \infty) \to \mathcal{M}(x_1, x_3),$$

which is a diffeomorphism onto its image. Moreover, we would like to argue that in the compactified moduli space $\overline{\mathcal{M}}(x_1, x_3)$, all the points near the image of $\mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3)$ is exactly obtained by gluing. For the following discussions, let $T \in [T_0, \infty)$.

(2a) Pregluing. Let's choose representatives $u_1 \in \mathcal{M}(x_1, x_2)$ and $u_2 \in \mathcal{M}(x_2, x_3)$. Equip u_1 with a positive cylindrical end near the puncture corresponding to x_2 and equip u_2 with a negative cylindrical end near the puncture corresponding to x_2.

Introduce the smooth cut-off function

$$\chi(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x \geq 1. \end{cases}$$

We can use it to "flatten" u_1 and u_2 by setting (here we use the coordinate over the cylindrical ends)

$$\overline{u}_1(s,t) = \begin{cases} u_1(s,t), & \text{if } s \leq T-1, \\ \exp_{x_2(t)}(\chi(T-s)\exp^{-1}_{x_2(t)}(u_1(s,t))), & \text{if } T-1 \leq s \leq T, \\ x_2(t), & \text{if } s \geq T, \end{cases}$$

$$\overline{u}_2(s,t) = \begin{cases} u_2(s,t), & \text{if } s \geq -T+1, \\ \exp_{x_2(t)}(\chi(T+s)\exp^{-1}_{x_2(t)}(u_2(s,t))), & \text{if } -T \leq s \leq -T+1, \\ x_2(t), & \text{if } s \leq -T. \end{cases}$$

Because $\overline{u}_1$ and $\overline{u}_2$ match with each other over the ends, we can define a new map over $\mathbb{R} \times S^1$ by introducing a neck-region $S^1 \times [0, 6T]$ of length $6T$:

$$\overline{u}(s,t) = \begin{cases} \overline{u}_1(s,t), & \text{if } s \leq T, \\ x_2(t), & \text{if } T \leq s \leq 5T, \\ \overline{u}_2(s-6T,t), & \text{if } s \geq 5T. \end{cases}$$

The map $\overline{u}$ is usually called the *pregluing* of u_1 and u_2.

Because the moduli spaces $\mathcal{M}(x_1, x_2)$ and $\mathcal{M}(x_2, x_3)$ are transversely cut out, as manifolds, they locally modeled on $\ker(D_{u_1})$ and $\ker(D_{u_2})$ respectively. When u_1 and u_2 vary, we use the following pregluing of kernel elements to characterize such variation. Take $\kappa_i \in \ker(D_{u_i})$, define

$$\overline{\kappa}(s,t) = \begin{cases} \kappa_1(s,t), & \text{if } s \leq T-1, \\ \Phi_{u_1(s,t)\to\overline{u}_1(s,t)}(\kappa_1(s,t)), & \text{if } T-1 \leq s \leq T, \\ \Phi_{u_1(s,t)\to\overline{u}_1(s,t)}(\kappa_1(s,t)) \cdot \chi(T+1-s), & \text{if } T \leq s \leq T+1, \\ 0, & \text{if } T+1 \leq s \leq 5T-1, \\ \Phi_{u_2(s-6T,t)\to\overline{u}_2(s,t)}(\kappa_2(s-6T,t)) \cdot \chi(s-5T+1), & \text{if } 5T-1 \leq s \leq 5T, \\ \Phi_{u_2(s-6T,t)\to\overline{u}_2(s,t)}(\kappa_2(s-6T,t)), & \text{if } 5T \leq s \leq 5T+1, \\ \kappa_2(s-6T,t), & \text{if } s \geq 5T+1. \end{cases}$$

Here the symbol Φ denotes the parallel transport. One can readily see that $\overline{\kappa}$ is a well-defined element of $\Gamma(\mathbb{R} \times S^1, \overline{u}^*TM)$. We will write the domain as Σ_T.

(2b) Functional spaces and linear recap.

Fix $\delta > 0$ whose with $|\delta|$ strictly less than the minimum of the absolute value of eigenvalues of the asymptotic operator of x_2, and fix $k \geq 4$. We will work with the weighted Sobolev space $W^{k,2,\delta}$, which combines the $W^{k,2}$-norm over the compact regions of a punctured (Σ, j) and the *weighted* $W^{k,2}$ norm (we write down the square)

$$\sum_{0\leq j\leq k} \int_{S^1\times[0,\infty)} |\nabla^j \xi(s,t)|^2 e^{2\delta s} ds \wedge dt \text{ or } \sum_{0\leq j\leq k} \int_{S^1\times(-\infty,0]} |\nabla^j \xi(s,t)|^2 e^{-2\delta s} ds \wedge dt$$

over the cylindrical ends together with

$$\sum_{0\le j\le k}\int_{S^1\times[0,6T]}|\nabla^j\xi(s,t)|^2e^{2\delta\min(s,6T-s)}ds\wedge dt$$

over the long-neck region $S^1\times[0,6T]$. Working with such spaces does not affect the index theory but will provide convenient estimates in the nonlinear theory.

Exercise 2.1. *Verify that using these norms, we have*

$$\|\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t))\|_{W^{k-1,2,\delta}}\le C\cdot e^{-\delta' T}(\|\kappa_1\|_{W^{k,2,\delta}}+\|\kappa_2\|_{W^{k,2,\delta}})$$

for any $0<\delta'<\delta$.

Going back to the preglued map $\overline{u}$, the Floer operator is written as

$$\mathcal{F}_{\overline{u}}(\xi)=\Phi_{\exp_{\overline{u}}(\xi)\to\overline{u}}\overline{\partial}_{H_t,J}(\exp_{\overline{u}}(\xi)):W^{k,2,\delta}(\Sigma_T,\overline{u}^*TM)\to W^{k-1,2,\delta}(\Sigma_T,\Omega^{0,1}_{\Sigma_T}\otimes\overline{u}^*TM).$$

In our discussion of index theories, we learned how to construct an approximate right inverse the linearized operator $D_{\overline{u}}$ after choosing bounded right inverses Q_{u_i} of D_{u_i}. Let's recall how it goes because here we need to incorporate the parallel transport.

(1) The first step is breaking. Given $\eta\in W^{k-1,2,\delta}(\Sigma_T,\Omega^{0,1}_{\Sigma_T}\otimes\overline{u}^*TM)$, we would like produce $\overline{\eta}_i\in W^{k-1,2,\delta}(\mathbb{R}\times S^1,\Omega^{0,1}_{\mathbb{R}\times S^1}\otimes\overline{u}_i^*TM)$ using cut-off functions. Define

$$\overline{\eta}_1(s,t)=\begin{cases}\eta(s,t), & \text{if } s\le 3T-1,\\ \chi(3T-s)\cdot\eta(s,t), & \text{if } 3T-1\le s\le 3T,\\ 0, & \text{if } s\ge 3T,\end{cases}$$

$$\overline{\eta}_2(s,t)=\begin{cases}0, & \text{if } s\le -3T,\\ \chi(3T+s)\cdot\eta(s+6T,t), & \text{if } -3T\le s\le -3T+1,\\ \eta(s+6T,t), & \text{if } s\ge -3T+1.\end{cases}$$

(2) Next, we use parallel transport to go from the flattened maps to the original maps u_i. So define

$$W^{k-1,2,\delta}(\mathbb{R}\times S^1,\Omega^{0,1}_{\mathbb{R}\times S^1}\otimes u_i^*TM)\ni\eta_i(s,t):=\Phi_{\overline{u}_i(s,t)\to u_i(s,t)}\overline{\eta}_i(s,t).$$

(3) Using the right inverses Q_{u_i}, we obtain elements

$$Q_{u_i}\eta_i\in W^{k,2,\delta}(\mathbb{R}\times S^1,u_i^*TM).$$

Note that we can use parallel transport to get from $Q_{u_1}\eta_1$ a section in $W^{k,2,\delta}(\mathbb{R}\times S^1,\overline{u}_i^*TM)$ supported over the bulk region union with the neck $S^1\times[0,6S]$, so is the case for $Q_{u_2}\eta_2$. We abuse the notation for the next bullet point.

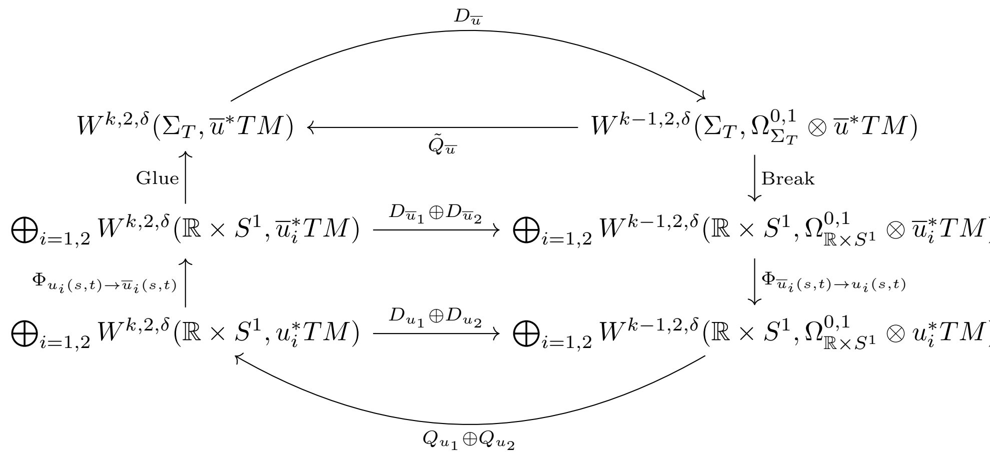
(4) Finally, we can glue the above two sections together to obtain

$$\text{Glue}(Q_{u_1}\eta_1,Q_{u_2}\eta_2)=\begin{cases}Q_{u_1}\eta_1, & \text{if } s\le 2T,\\ \chi(4T-s)\cdot Q_{u_1}\eta_1+\chi(s-2T)Q_{u_2}\eta_2(s-6T,t), & \text{if } 2T\le s\le 4T,\\ Q_{u_2}\eta_2(s-6T,t), & \text{if } s\ge 4T.\end{cases}$$

Combining these steps together, we obtain

$$\tilde{Q}_{\overline{u}} : W^{k-1,2,\delta}(\Sigma_T, \Omega^{0,1}_{\Sigma_T} \otimes \overline{u}^*TM) \to W^{k,2,\delta}(\Sigma_T, \overline{u}^*TM)$$

We can summarize the above construction using the following diagram



Lemma 2.2. *As an operator, $\tilde{Q}_{\overline{u}}$ satisfies*

$$\|\tilde{Q}_{\overline{u}}\| \le C, \qquad \|D_{\overline{u}}\tilde{Q}_{\overline{u}} - 1\| \le \frac{1}{2}$$

for T sufficiently large.

Proof. The only difference between the current setting and our discussion in the linear case is the presence of parallel transport in the discussion, but the error introduced here can be effectively controlled by the exponential decay behavior: $u_1(s,t) \to x_2(t)$ as $s \to \infty$ exponentially fast with decay rate $> \delta$, and similarly for $u_2(s,t)$ as $s \to -\infty$.

The boundedness $\tilde{Q}_{\overline{u}}$ is a straightforward consequence of the construction. For the rest, to be more precise, what needs to be proved can be reduced to the following sequence of statements.

(1) For the bottom square, we can compute

$$\begin{aligned} &\|(D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \big(\Phi_{u_i\to\overline{u}_i} \circ (Q_{u_1} \oplus Q_{u_2}) \circ \Phi_{\overline{u}_i\to u_i}\big) - 1\| \\ =& \|(D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \big(\Phi_{u_i\to\overline{u}_i} \circ (Q_{u_1} \oplus Q_{u_2}) \circ \Phi_{\overline{u}_i\to u_i}\big) - \Phi_{u_i\to\overline{u}_i} \circ \big(D_{u_1} \oplus D_{u_2}\big) \circ (Q_{u_1} \oplus Q_{u_2})\big)\Phi_{\overline{u}_i\to u_i}\| \\ \le& \|(Q_{u_1} \oplus Q_{u_2})\big)\Phi_{\overline{u}_i\to u_i}\| \cdot \|(D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \Phi_{u_i\to\overline{u}_i} - \Phi_{u_i\to\overline{u}_i} \circ \big(D_{u_1} \oplus D_{u_2}\big)\|. \end{aligned}$$

The first operator in is bounded by our assumption. As for the second operator, one can spell out the explicit formula of the linearized operators as what we did before. It turns out that the operator norm is bounded from above by a constant multiple of the C^k-distance of u_i and $\overline{u}_i$, which is in turned bounded by $C \cdot e^{-\delta' T}$ for some $\delta' > 0$.

(2) For the top square, we wish to show that

$$\|D_{\overline{u}} - \text{Glue} \circ (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \text{Break}\|$$

is sufficiently close to 0. This is the situation we encounter in the discussion of linear gluing. The point is that our Sobolev weights and the support of the cut-off functions have been chosen very nicely so that is again bounded by $C \cdot e^{-\delta' T}$.

(3) Combining the above two steps, we have

$$\begin{aligned}
&\|D_{\overline{u}} \circ \tilde{Q}_{\overline{u}} - 1\| \\
\leq\ &\|D_{\overline{u}} \circ \tilde{Q}_{\overline{u}} - \mathrm{Glue} \circ (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \mathrm{Break} \circ Q_{\overline{u}}\| + \|\mathrm{Glue} \circ (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \mathrm{Break} \circ \tilde{Q}_{\overline{u}} - 1\| \\
\leq\ &C \cdot \|D_{\overline{u}} - \mathrm{Glue} \circ (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \mathrm{Break}\| + \|\mathrm{Glue} \circ (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \mathrm{Break} \circ \tilde{Q}_{\overline{u}} - (D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \\
&\circ \big(\Phi_{u_i \to \overline{u}_i} \circ (Q_{u_1} \oplus Q_{u_2}) \circ \Phi_{\overline{u}_i \to u_i}\big)\| + \|(D_{\overline{u}_1} \oplus D_{\overline{u}_2}) \circ \big(\Phi_{u_i \to \overline{u}_i} \circ (Q_{u_1} \oplus Q_{u_2}) \circ \Phi_{\overline{u}_i \to u_i}\big) - 1\|.
\end{aligned}$$

The middle term can be estimated by tracing through the definition.

Then one sees that for $T \gg 1$, we have the desired bounds. □

The upshot is, we can define the bounded genuinely right inverse $Q_{\overline{u}}$ of $D_{\overline{u}}$ using

$$\tilde{Q}_{\overline{u}} \circ (D_{\overline{u}} \tilde{Q}_{\overline{u}})^{-1} = \tilde{Q}_{\overline{u}} \circ (1 - (1 - D_{\overline{u}} \tilde{Q}_{\overline{u}}))^{-1} = \tilde{Q}_{\overline{u}} \circ \sum_{k \geq 0} (1 - D_{\overline{u}} \tilde{Q}_{\overline{u}})^k.$$

(2c) Gluing map and Newton–Picard iteration.

We wish to perturb the map $\exp_{\overline{u}(s,t)}(\overline{\kappa}(s,t))$ by an element in $\mathrm{im}(Q_{\overline{u}})$ to obtain a genuine solution to the Floer equation. In other words, we would like to solve the equation

$$\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}} \eta) = 0$$

for $\eta \in W^{k-1,2,\delta}(\Sigma_T, \Omega^{0,1}_{\Sigma_T} \otimes \overline{u}^* TM)$. The following discussion is useful for thinking about this equation. Using the linearization $D_{\overline{u}}$, we can write the Taylor expansion of $\mathcal{F}_{\overline{u}}$ as

$$\mathcal{F}_{\overline{u}} = D_{\overline{u}} + \text{quadratic terms},$$

which can be formulated precisely as follows.

Lemma 2.3. *Given $\xi_1, \xi_2 \in W^{k,2,\delta}(\Sigma_T, \overline{u}^* TM)$, we have*

$$\|D_{\overline{u}}(\xi_1 - \xi_2) - (\mathcal{F}_{\overline{u}}(\xi_1) - \mathcal{F}_{\overline{u}}(\xi_2))\|_{W^{k-1,2,\delta}} \leq C \cdot \|\xi_1 - \xi_2\|_{W^{k,2,\delta}} \cdot (\|\xi_1\|_{W^{k,2,\delta}} + \|\xi_2\|_{W^{k,2,\delta}}). \quad \square$$

For the proof, one can look at [MS04, Proposition 3.5.3]. The result is, of course, expected from the point of view of Taylor expansions.

Now we can use the previous preparation to define the gluing map. This process is an explicit implementation of the contraction mapping principle as in the proof of implicit function theorems, usually referred to as the *Newton–Picard iteration*.

Proposition 2.4. *There exists a $C' > 0$ such that for $\|\kappa_i\|$ sufficiently small, the equation*

$$\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}} \eta) = 0$$

with the constraint $\|\eta\| \leq C'$ admits a unique solution.

Proof. Note that the equation we would like to solve is equivalent to

$$\eta - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta) = \eta,$$

in other words, the fixed point of the mapping

$$\eta \mapsto \eta - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta).$$

Set $\xi_1 = \overline{\kappa}(s,t) + Q_{\overline{u}}\eta_1$ and $\xi_2 = \overline{\kappa}(s,t) + Q_{\overline{u}}\eta_2$ in the quadratic estimate, we see that

$$\|\eta_1 - \eta_2 - (\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta_1) - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta_2)))\| \leq 2C \cdot \|Q_{\overline{u}}\| \cdot \|\eta_1 - \eta_2\| \cdot (\|\overline{\kappa}(s,t)\| + \|Q_{\overline{u}}\eta_1\| + \|Q_{\overline{u}}\eta_2\|).$$

Therefore, we see that we can choose C' such that for $\|\kappa_i\|$ sufficiently small we necessarily have

$$\|\eta_1 - \eta_2 - (\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta_1) - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta_2)))\| \leq \frac{1}{2}\|\eta_1 - \eta_2\|$$

when $\|\eta_i\| \leq C'$. Moreover, the condition $\|\eta\| \leq C'$ can be assumed to be preserved under such a contraction mapping. This is true because the quadratic estimates applied to $\xi_1 = \overline{\kappa}(s,t) + Q_{\overline{u}}\eta$ and $\xi_2 = \overline{\kappa}(s,t)$ gives us

$$\|\eta - (\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta) - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t)))\| \leq 2C \cdot \|Q_{\overline{u}}\eta\| \cdot (\|Q_{\overline{u}}\eta\| + \|\overline{\kappa}(s,t)\|),$$

the estimates $\|\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t)))\| \leq C \cdot (\|\kappa_1\| + \|\kappa_2\|)$, and the quadratic term $\|Q_{\overline{u}}\eta\|^2$.

Then the iteration is given by

$$\eta_0 := \overline{\kappa},$$
$$\eta_m := \eta_{m-1} - \mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta_{m-1}).$$

This is a Cauchy sequence, which admits a limit in the complete metric space, the C' closed ball in $W^{k-1,2,\delta}(\Sigma_T, \Omega^{0,1}_{\Sigma_T} \otimes \overline{u}^*TM)$. The limit is the defined to be η, and it solved the genuine Floer equation by construction. Uniqueness is straightforward: $\|\eta - \eta'\| \leq \frac{1}{2}\|\eta - \eta'\|$ if η and η' are fixed points. □

To sum up, given $(u_1, u_2, T) \in \mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3) \times [T_0, \infty)$, together with κ_1 and κ_2 which are elements in the local coordinate chart of u_1 and u_2 respectively defined from the kernel of the linearized operator, we find a solution to the Floer equation

$$\mathcal{F}_{\overline{u}}(\overline{\kappa}(s,t) + Q_{\overline{u}}\eta) = 0$$

from the Newton–Picard iteration. This is the gluing map

$$(\kappa_1, \kappa_2, T) \in \ker(D_{u_1}) \times \ker(D_{u_2}) \times [T_0, \infty) \to \mathcal{M}(x_1, x_3).$$

Note that for κ_i sufficiently small, $\mathcal{F}_{u_i}(\kappa_i)$ is very close to 0. Using the right inverses Q_{u_i}, one can then use the same idea to find a unique $\eta_i \in W^{k-1,2,\delta}(\mathbb{R} \times S^1, \Omega^{0,1}_{\mathbb{R}\times S^1} \otimes u_i^*TM)$ such that

$$\mathcal{F}_{u_i}(\kappa_i + Q_{u_i}\eta_i) = 0.$$

This produces maps

$$\ker(D_{u_1}) \to \mathcal{M}(x_1, x_2), \qquad \ker(D_{u_2}) \to \mathcal{M}(x_2, x_3)$$

which is a diffeomorphism onto its image. This should be thought of as the "exponential map" on the moduli spaces. Using the exponential map as local coordinates, we can view the gluing map as

$$\mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3) \times [T_0, \infty) \to \mathcal{M}(x_1, x_3).$$

(2d) Properties of the gluing map.

We wish to show that the gluing map *completely* describe the boundary of the compactified moduli space $\overline{\mathcal{M}}(x_1, x_3)$. In other words, we would like to see that the gluing map is smooth, injective, and surjective.

Exercise 2.5. *Trace through the construction to show that the gluing map is smooth. This can be formally summarized as the* smooth dependence *of the implicit functions on parameters.*

For the injectivity, note that the neck length of two glued maps with the same image should agree. Suppose that we have $(\kappa_1, \kappa_2) \in \ker(D_{u_1}) \times \ker(D_{u_2})$ and $(\kappa_1', \kappa_2') \in \ker(D_{u_1}) \times \ker(D_{u_2})$ such that

$$\exp_{\overline{u}}(\kappa + Q_{\overline{u}}\eta) = \exp_{\overline{u}}(\kappa' + Q_{\overline{u}}\eta').$$

See we see that

$$\kappa + Q_{\overline{u}}\eta = \kappa' + Q_{\overline{u}}\eta',$$

to which we can apply $D_{\overline{u}}$ to see that $\eta = \eta'$, so we know that $\kappa = \kappa'$. Then the agreement of (κ_1, κ_2) and (κ_1', κ_2') follows from the linear pregluing and unique continuation.

For surjectivity, let's assume that we are given $u_\nu : \Sigma_{T_\nu} \to M$ in $\mathcal{M}(x_1, x_3)$ with $T_\nu \to \infty$ as $\nu \to \infty$ such that u_ν converges to the broken trajectory formed by u_1 and u_2. We want to show that for ν sufficiently large, any u_ν comes from gluing. Then using the preglued map $\overline{u}^\nu$ from the flattened maps $\overline{u}_1^\nu$ and $\overline{u}_2^\nu$ with respect to the parameter T_ν, we can define ξ_ν by the formula

$$\exp_{\overline{u}^\nu}(\xi_\nu) = u_\nu.$$

Then we know that $\|\xi_\nu\|_{C^0} \to 0$ as $\nu \to \infty$. We then break the discussion into two parts.

(1) Over the "bulk region" of the glued surface Σ_{T_ν}, i.e., away from the neck region $S^1 \times [2, 6T_\nu - 2]$, we necessarily have u_ν converges uniformly in all of its derives over compact subsets to u_1 and u_2 in the respective region. This is true by the nature of Gromov–Floer compactness. Moreover, the sequence ξ_ν converges uniformly in all of its derives over compact subsets to 0.

(2) Over the neck region $S^1 \times [T_\nu, 5T_\nu]$, let's observe that ξ_ν can also be obtained via the equation

$$\exp_{x_2(t)}(\xi_\nu(s,t)) = u_\nu(s,t)$$

because $\overline{u}^\nu$ is flattened to be the Hamiltonian orbit $x_2(t)$. We further define $\tilde{\xi}_\nu$ over the neck region $S^1 \times [0, 6T_\nu]$ using

$$\exp_{x_2(t)}(\tilde{\xi}_\nu(s,t)) = u_\nu(s,t).$$

Then [Sal99, Lemma 2.11] can be modified to prove that over $1 \leq s \leq N-1$, we have

$$|\nabla^k \tilde{\xi}_\nu(s,t)| \leq C_k(e^{-\delta'' s}(\int_{S^1} |\tilde{\xi}(0,t)|^2)^{1/2} + e^{-\delta''(6T_\nu - s)}(\int_{S^1} |\tilde{\xi}(6T_\nu,t)|^2)^{1/2}).$$

Therefore, we see that $\xi_\nu \to 0$ in the $W^{k,2,\delta}$-norm. We can write

$$\xi_\nu = \kappa_\nu + Q_{\overline{u}^\nu}\eta_\nu$$

where $\kappa_\nu \in \ker(D_{u_\nu})$. Then for ν sufficiently large, if we apply the Newton–Picard iteration to κ_ν over the preglued curve $\overline{u}^\nu$, the η solved there is necessarily η_ν due to the uniqueness, which we can use because κ_ν has norm sufficiently close to 0.

References

[MS04] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications, vol. 52, American Mathematical Society, 2004.

[Sal99] Dietmar Salamon, *Lectures on Floer homology*, Symplectic geometry and topology (Park City, UT, 1997), IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, pp. 143–229.