Topic 3: compactness

1. Energy inequalities

Let (Σ, j) be a punctured Riemann surface equipped with cylindrical ends. Still, (M, ω) is a compact symplectic manifold and J is an ω-compatible almost complex structure.

(1a) Inhomogeneous Cauchy–Riemann equations. The space $C^\infty(M)$ is a Lie algebra under the Poisson bracket

$$(f, g) \mapsto \{f, g\} := L_{X_f} g = \omega(X_f, X_g),$$

which is the Lie algebra of the group of Hamiltonian diffeomorphisms. We can consider the space of Hamiltonian connections, which is modeled on

$$\Omega^1(\Sigma, C^\infty(M)),$$

1-forms valued in the Lie algebra $C^\infty(M)$. Once choosing $\alpha \in \Omega^1(\Sigma, C^\infty(M))$, we can write down a Cauchy–Riemann type equation

$$(du - X_\alpha(u))^{0,1}_J = 0.$$

It means the following. Given such α, if we choose local coordinates $(s, t) \in U \subset \Sigma$, we can write $\alpha = F ds + G dt$, where $F, G \in C^\infty(U, C^\infty(M))$. Then using the relation $dH = \omega(-, X_H)$, we can define the 1-form over Σ taking value in $\mathfrak{X}(M)$, the space of vector fields on M,

$$X_\alpha = X_F ds + X_G dt.$$

Then X_α can be viewed as a vector bundle homomorphism of

$$\begin{array}{ccc} (\pi_\Sigma^* T\Sigma, \pi_\Sigma^* j) & \longrightarrow & (\pi_M^* TM, \pi_M^* J) \\ & \searrow \quad \swarrow & \\ & \Sigma \times M, & \end{array}$$

so is du. Then the notation $(-)^{0,1}_J$ is the projection to the complex anti-linear part

$$- \mapsto \frac{1}{2}(- + J \circ - \circ j).$$

Important special cases of this equation include the following.

(1) When $\alpha \equiv 0$, we have the standard J-holomorphic map equation

$$du \circ j = J \circ du.$$

(2) If $\Sigma = \mathbb{R} \times S^1$ and the $\alpha = H_t dt$, which is pulled back via the projection $\mathbb{R} \times S^1 \to S^1$, because $X_\alpha = X_{H_t} dt$, we obtain the Floer equation

$$\partial_s u + J(u)(\partial_t - X_{H_t}(u)) = 0.$$

(3) If $\Sigma = \mathbb{R} \times S^1$ and the ds-component of α is 0, i.e., $\alpha = H_{s,t}dt$, we get the continuation equation
$$\partial_s u + J(u)(\partial_t - X_{H_{s,t}}(u)) = 0.$$

(4) In general, because we are supposed to use $(du - X_\alpha(u))_J^{0,1} = 0$ to define operations in Floer theory, we require the following constraint. Suppose $\epsilon : [0, +\infty) \to \Sigma$ or $(-\infty, 0] \times S^1 \to \Sigma$ is either a positive or negative cylindrical end. Then we ask
$$\epsilon^* \alpha = H_t dt$$
for some 1-periodic Hamiltonian $H_t \in C^\infty(S^1 \times M, \mathbb{R})$. In particular, the equation is reduced to the Floer equation over the cylindrical ends.

(1b) Energy estimate. Now consider the product $\Sigma \times M$. Then for any $\alpha \in \Omega^1(\Sigma, C^\infty(M))$, we can define $\tilde{\alpha} \in \Omega^1(\Sigma \times M)$, whose value as a linear functional at the tangent space of $(z, x) \in \Sigma \times M$ takes (ξ, η) to $\alpha_z(\xi)(x)$. Then we can define a 2-form on $\Sigma \times M$
$$\omega_\alpha := \pi_M^* \omega - d\alpha.$$

Exercise 1.1. *Show that under the local coordinate $\alpha = Fds + Gdt$, the 2-form ω_α can be written as*
$$\omega - d_M F \wedge ds - d_M G \wedge dt + (\partial_t F - \partial_s G) ds \wedge dt,$$
where d_M denotes the de Rham differential along the M-direction.

The 2-form ω_α defines a connection of the projection $\pi_\Sigma : \Sigma \times M \to \Sigma$ as follows. We declare the horizontal space at $(z, x) \in \Sigma \times M$ to be
$$\{(\xi', \eta') \in T_{(z,x)}(\Sigma \times M) \mid \forall (\xi, \eta) \in \ker(d\pi_M)_{(z,x)}, \text{ we have } \omega_\alpha\big((\xi, \eta), (\xi', \eta')\big) = 0\},$$
i.e., the symplectic orthogonal complement of the vertical tangent space.

Lemma 1.2. *The connection associated with ω_α has curvature F_α given by*
$$(\partial_s G - \partial_t F + \{F, G\}) ds \wedge dt$$
using the local coordinates $\alpha = Fds + Gdt$.

Proof. One can see that the horizontal lifts of ∂_s and ∂_t are respectively
$$\partial_s + X_F, \qquad \partial_t + X_G.$$
Then the projection of $[\partial_s + X_F, \partial_t + X_G]$ to the vertical direction is
$$X_{\partial_s G} - X_{\partial_t F} + X_{\{F,G\}}.$$
Translating back to the Lie algebra $C^\infty(M)$, we obtain the desired formula. □

Definition 1.3. *The* geometric/analytic *energy of a solution to $(du - X_\alpha(u))_J^{0,1} = 0$ is defined to be*
$$E^{geom}(u) := \frac{1}{2} \int_\Sigma |du - X_\alpha(u)|^2 \mathrm{dvol}_\Sigma.$$

Proposition 1.4. *For $u : \Sigma \to M$ satisfying $(du - X_\alpha(u))_J^{0,1} = 0$, we have*

$$E^{geom}(u) = \int_\Sigma u^*\omega + \int_\Sigma F_\alpha(u).$$

Proof. Let's perform some calculations. In the local coordinates (s,t), the equation can be written as

$$\big(\partial_s u - X_F(u)\big) + J(u)\big(\partial_t - X_G(u)\big) = 0.$$

Therefore, treating ds and dt as orthonormal basis, we have

$$\begin{aligned}
&\frac{1}{2}|du - X_\alpha(u)|^2 d\mathrm{vol}_\Sigma \\
&= \frac{1}{2}\big(g_J(\partial_s u - X_F(u), \partial_s u - X_F(u)) + g_J(\partial_t u - X_G(u), \partial_t u - X_G(u)\big)ds \wedge dt \\
&= \frac{1}{2}\big(\omega(\partial_s u - X_F(u), J(u)(\partial_s u - X_F(u))) + \omega(\partial_t u - X_G(u), J(u)(\partial_t u - X_G(u))\big)ds \wedge dt \\
&= \omega(\partial_s u - X_F(u), \partial_t u - X_G(u))ds \wedge dt \\
&= u^*\omega + F_\alpha(u).
\end{aligned}$$

In the third line, we use the compatibility $g(-,-) = \omega(-, J-)$ in the fourth line, we use the equation satisfied by u; the last line is derived by tracing through the definition. □

In contrast with the geometric energy, we can define *topological energy* $E^{top}(u)$ in the following context. All the definitions satisfy the important estimate

$$E^{geom}(u) \leq E^{top}(u).$$

(1) If $\alpha \equiv 0$ and Σ is a closed Riemann surface such that $u_*([\Sigma]) = A \in H_2(M, \mathbb{Z})$, then

$$E^{geom}(u) = \int_\Sigma u^*\omega = \omega(A) =: E^{top}(u).$$

That is to say, after prescribing the homology class of a J-holomorphic map, $E^{geom}(u)$ is uniformaly bounded from above by $\omega(A)$.

(2) For the Floer equation, note that we have

$$E^{geom}(u) = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_{H_t}(u))ds \wedge dt = \int_{\mathbb{R}\times S^1} u^*\omega - \int_{\mathbb{R}\times S^1} dH_t(u)(\partial_s u)ds \wedge dt.$$

Because $dH_t(u)(\partial_s u)ds \wedge dt = d(H_t(u)) \wedge dt$, if $\lim_{s\to\pm\infty} u(s,t) = x^\pm(t)$, we see that the last integral becomes

$$\int_{S^1} H_t(x^+(t))dt - \int_{S^1} H_t(x^-(t))dt,$$

so we have

$$0 \leq E^{geom}(u) = \int_{\mathbb{R}\times S^1} u^*\omega - \int_{S^1} H_t(x^+(t))dt + \int_{S^1} H_t(x^-(t))dt.$$

Furthermore, if the orbits $x^\pm$ are capped by $u^\pm : D^2 \to M$, the above formula is further reduced to

$$\int_{D^2} (u^+)^*\omega - \int_{S^1} H_t(x^+(t))dt - (\int_{D^2} (u^-)^*\omega - \int_{S^1} H_t(x^-(t))dt),$$

which is the difference of the action functional $-\mathcal{A}([x^+,u^+])+\mathcal{A}([x^-,u^-])$. Thus, under our convention, the Floer differential does not decrease the energy. In this case, we declare $E^{top}=-\mathcal{A}([x^+,u^+])+\mathcal{A}([x^-,u^-])$.

(3) For the continuation map equation defined using $\alpha = H_{s,t}dt$ with asymptotics $\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t)$ (capped) Hamiltonian orbits of $H_t^{\pm}$, we have

$$E^{geom}(u) = -\mathcal{A}([x^+,u^+])+\mathcal{A}([x^-,u^-]) + \int_{\mathbb{R}\times S^1} \partial_s H_{s,t}(u) ds\wedge dt.$$

Therefore, if we ask for the pointwise condition $\partial_s H_{s,t}\leq 0$, we see that

$$E^{geom}(u) \leq -\mathcal{A}([x^+,u^+])+\mathcal{A}([x^-,u^-]) =: E^{top}(u),$$

which can be used to define continuation maps preserving the energy filtration. More generally, the pointwise condition

$$\frac{F_\alpha}{ds\wedge dt}\leq 0$$

ensures that $E^{geom}(u)$ is bounded from above by the weighted sum of the energy of the action functional at the critical points.

2. Useful results for compactness

The general question is the following: given a sequence of pseudo-holomorphic maps $u_k:\Sigma_k\to M$, how to ensure that a limit exists in a suitable sense? Note that this covers the setting of Floer equations because of Gromov's graph trick. We collect some useful analytic results.

(2a) C^∞_{loc}-convergence. The following statement is deduced from elliptic bootstrapping technique, which also appears in other contexts like gauge theory.

Theorem 2.1. *Let (M,J) be an almost complex manifold. Suppose $J_k\to J$ is a sequence of almost complex structures which converges to J in the C^∞ topology. Then given a Riemann surface (Σ,j) with smooth maps*

$$u_k:\Sigma\supset\Omega_k\to M$$

defined over an increasing sequence of precompact open subsets Ω_k exhausting Σ satisfying

$$du_k\circ j = J_k\circ du$$

and the uniform bound

$$\sup_k |du_k| < \infty$$

over any compact subset $K\subset\Sigma$, we can find a smooth map

$$u:\Sigma\to M, \qquad du\circ j = J\circ du$$

such that $\{u_k\}$ admits a subsequence which converges uniformly in all derivatives to u over compact subsets of Σ.

In other words, uniform control on the derivative ensures sequential compactness.

Proof. This is a typical application of the elliptic bootstrapping technique. We first show that for any $u : (D^2, j) \to (M, J)$ satisfying

$$\partial_s u + J(u)\partial_t u = 0,$$

we have $|D^k u(0)| \leq C \cdot \|u\|_{C^1}$ for some C depending on M, J, and k. Indeed, we can apply the operator $\partial_s - \partial_t J(u)$ to the J-holomorphic map equation to obtain

$$u_{ss} + u_{tt} = \dot{J}(u, u_t)u_s - \dot{J}(u, u_s)u_t.$$

As the RHS has L^2-norm bounded from above by $\|u\|_{C^1}$, standard properties of the Laplacian operator implies that $\|u\|_{W^{2,2}(K)} \leq C \cdot \|u\|_{C^1}$ for any compact subset K of the open 2-disc D°. Using the pointwise bound on $\dot{J}$ due to the assumption that $\|du\|_{C^0}$ is bounded, we deduce that over any such K, the RHS has $W^{1,2}$-norm bounded from above by $C \cdot \|u\|_{C^1}$. Because of this, we can apply the elliptic regularity of the Laplacian operator again to conclude that over any $K \subset D^\circ$, we have $\|u\|_{W^{3,2}} \leq C \cdot \|u\|_{C^1}$. Then use such $W^{3,2}$-bound, we see that the RHS has $W^{2,2}$-bound controlled by a constant multiple of $\|u\|_{C^1}$, which then implies the $W^{4,2}$-bound on u on compact subsets of D°. Iterating, we see that for any $k \geq 1$, there exists a constant $C = C(k, M, J)$ such that

$$\|u\|_{W^{k,2}(K)} \leq C \cdot \|u\|_{C^1}$$

for any compact $K \subset D^\circ$. Using the Sobolev embedding $C^k \hookrightarrow W^{k',2}$ for suitable k', we obtain the desired estimate by specializing to the point 0. [1]

Going back to the situation we have. Inspecting the proof, we see that the constant C in the final estimate depends continuously on the first $k+2$ derivatives of J. Therefore, for any compact subset $K \subset \Sigma$, upon passing to a subsequence, we see that for any $m \geq 0$, we have the uniform bound

$$\|u_k\|_{C^m} \leq C \cdot \sup_k |du_k|$$

where C depends on M, J, and m. Using the Arzelà–Ascoli theorem, and combining the diagonal argument with the exhausting family of open subsets of Σ, after passing to a subsequence, we can find $u : \Sigma \to M$ such that $u_k \to u$ uniformly in all derivatives over compact subsets of Σ. It follows from the construction that u satisfies $du \circ j = J \circ du$. □

(2b) Bubbling. From the discussion on C^∞_{loc}-convergence, we see that sequential compactness of J-holomorphic maps may fail if the gradient blows up.

One typical example is the following. Let's look at the holomorphic $u : D^2 \to \mathbb{C}P^1$ defined by embedding the unit disc. We endow $\mathbb{C}P^1$ with the Fubini–Study metric. Then consider $u_N : D^2 \xrightarrow{u} D^2 \xrightarrow{\times N} \mathbb{C} \subset \mathbb{C}P^1$. We see that the geometric energy of u_N is the same as $E^{geom}(u)$. However, the gradient of u_N blows up at 0 for obvious reasons. On the other hand, we can rescale the domain of u_N by $D^2_N \xrightarrow{\times 1/N} D^2$, a conformal reparametrization. Then the rescaled map converges to the identity map $\mathbb{C}P^1 \to \mathbb{C}P^1$ except at ∞, while the original sequence $\{u_N\}$ converges to ∞ away from $0 \in D^2$. In other words, we should think the limit as $u : \mathbb{C}P^1 \wedge D^2 \to \mathbb{C}P^1$, which is the identity map over $\mathbb{C}P^1$ and the constant map to ∞ over D^2.

[1] The same argument can be applied to the Sobolev spaces $W^{k,p}$ using Calderon–Zygmund estimates.

Exercise 2.2. *Consider holomorphic maps $\mathbb{C}P^1 \to \mathbb{C}P^1$, which can be presented as a rational function*

$$z \mapsto u(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials in z. Then the degree of the map is simply the maximum of the degrees of $p(z)$ and $q(z)$, prescribing the topological and geometric energy.

(1) *Equipping $\mathbb{C}P^1$ with the Fubini–Study metric $(1+x^2+y^2)^{-2}dx \wedge dy$, show that the norm of the derivative of $u(z)$ satisfies*

$$|du(z)|_{\mathrm{FS}} = \sqrt{2}|u'(z)|\frac{1+|z|^2}{1+|u(z)|^2}.$$

(2) *Consider the sequence of rational maps*

$$u_k(z) = \frac{zp(z)}{(z-a_k)q(z)}$$

for some degree k polynomials $p(z)$ and $q(z)$ with $p(0) = q(0) = 1$ which are coprime to each other. Moreover, assume that $a_k \to 0$. Show that $du_k(0)$ diverges but u_k converges uniformly to a holomorphic map on $\mathbb{C}P^1 \setminus \{0\}$. Construct reparametrizations of u_k so that the new sequence converges to a degree 1 rational function.

Next, we provide a general discussion of these so-called *bubbling-off* analysis.

Lemma 2.3 (Hofer's lemma)**.** *Let (X, d) be a complete metric space, let $f : X \to \mathbb{R}_{\geq 0}$ be locally bounded, and let $M < \infty$. For every $p_0 \in X$, there exists $p \in X$ with $f(p) \geq f(p_0)$ and $d(p, p_0) \leq 2Mf(p_0)^{-1}$ such that $d(x, p) \leq M \cdot f(p)^{-1} \Rightarrow f(x) \leq 2f(p)$.*

Proof. Let's argue by contradiction. Suppose that p_0 itself does not satisfy the said property. Then we can find p_1 such that $d(p_0, p_1) \leq M \cdot f(p_0)^{-1}$ and $f(p_1) > 2f(p_0)$. If p_1 still violates the desired property, then we can find p_2 with $d(p_1, p_2) \leq M \cdot f(p_1)^{-1}$ and $f(p_2) > 2f(p_1)$. Iterating, we see that for any $k \geq 1$, we will find p_k with $d(p_{k-i}, p_k)$ uniformly bounded by $C \cdot 2^{-i}$ but $f(p_k) \geq 2^k f(p_0)$. By the completeness of X, we see that $p_k \to p$ by passing to a subsequence but $f(p) = \infty$, contradicting the boundedness assumption on f. □

Hofer's lemma can be applied in the following way. Suppose we have a sequence of J-holomorphic maps $u_k : (\Sigma, j) \to (M, J)$ and a sequence of points $\{z_k\} \subset \Sigma$ such that $|du(z_k)| \to \infty$. For each k, let $f(z) = |du(z)|$ and let $p_0 = z_k$. Setting $M = |du(z_k)|^{1/2}$. Then we can find z'_k, which is the p in Hofer's lemma, such that for any $z \in \Sigma$ with $d(z, z'_k) \leq 1/\sqrt{|du(z'_k)|}$ we have $|du(z)| \leq 2|du(z'_k)|$. Choosing local coordinates around z'_k, we can consider the reparametrized map

$$\tilde{u}_k := z \mapsto u(\frac{z}{|du(z'_k)|} + z'_k)$$

defined over the ball in $\mathbb{C}$ centered at 0 with radius $\sqrt{|du(z'_k)|}$. Then $\tilde{u}_k$ satisfies $|d\tilde{u}_k| \leq 2$, so we see that after passing to a subsequence, $\tilde{u}_k$ converges to a holomorphic plane $\tilde{u} : \mathbb{C} \to M$.

(2c) Convergence modulo bubbling. Combining the discussion in the previous two subsections, we can state the key compactness result.

Theorem 2.4. *Let (M, J) be an almost complex manifold. Suppose $J_k \to J$ is a sequence of almost complex structures which converges to J in the C^∞ topology. Let (Σ, j) be a Riemann surface with smooth maps*

$$u_k : \Sigma \supset \Omega_k \to M$$

defined over an increasing sequence of precompact open subsets Ω_k exhausting Σ satisfying

$$du_k \circ j = J_k \circ du$$

and the uniform bound on the geometric energy

$$\frac{1}{2}\int_{\Omega_k} |du_k|^2 \text{dvol}_{\Omega_k} < \infty$$

We can find a smooth map

$$u : \Sigma \to M, \qquad du \circ j = J \circ du$$

and a finite set of points $Z = \{z_1, \ldots, z_l\} \subset \Sigma$ such that

(1) *$\{u_k\}$ admits a subsequence (written using the same notation) which converges uniformly in all derivatives to u over compact subsets of $\Sigma \setminus Z$.*
(2) *For every j and $\epsilon > 0$ the limit*

$$m(z_j, \epsilon) := \lim_{k\to\infty} E(u_k, B_\epsilon(z_j)) := \lim_{k\to\infty} \frac{1}{2}\int_{B_\epsilon(z_j)} |du_k|^2 \text{dvol}$$

exists and is a continuous function in ϵ. Moreover, $m(z_j) := \lim_{\epsilon\to 0} m(z_j, \epsilon)$ is uniformly bounded from below by a constant $\hbar$ depending only on (M, J).
(3) *We have the conservation of energy*

$$\lim_k E^{geom}(u_k) = E^{geom}(u) + \sum_j m(z_j).$$

For the full proof, please consult [MS04, Chapter 4]. We only list some ingredients without providing the full proof.

(1) For a holomorphic plane $u : \mathbb{C} \to (M, J)$ constructed from the bubbling analysis, using the reparametrization $z \mapsto 1/z$, investigating u near ∞ follows into the investigations of holomorphic $u : D^2 \setminus \{0\} \to M$ with bounded energy. The following is known as the *Removal Singularity Theorem.*

Theorem 2.5. *If $u : D^2\setminus\{0\} \to (M, J)$ has bounded geometric energy and compact image after taking closure (to deal with when M is noncompact), then u extends smoothly across 0 to be a J-holomorphic map on the whole D^2.*

Accordingly, the J-holomorphic planes are actually extendable to J-holomorphic spheres, explaining the name *bubbles.*

(2) The lower bound $m(z_j) \geq \hbar$ is known as *energy quantization.* It is usually deduced from the so-called *mean-value inequality*, a form of *ϵ-regularity.*

Theorem 2.6. *Given an almost complex manifold (M, J), there exists $\delta > 0$ and $C > 0$ such that for any J-holomorphic $u : D_r^2 \to (M, J)$, we have*

$$\int_{D_r^2} |du|^2 d\text{vol} \leq \delta \qquad \Rightarrow \qquad |du(0)| \leq \frac{C}{r^2} \int_{D_r^2} |du|^2 d\text{vol}.$$

We can prove the energy quantization from here very easily. Assume that for (M, J), for any $\epsilon > 0$, we can find $u : S^2 \to M$ non-constant which is J-holomorphic and $E^{geom}(u) \leq \epsilon$. For $\epsilon < \delta$, we can choose D_r to be the open ball in $\mathbb{C} \subset S^2$. Since we are free to choose the origin and r can be arbitratily large, we see that $du \equiv 0$, contradicting the assumption that u is not a constant.

(3) Using energy quantization, it is easy to see that there are only finitely many bubbles assuming the bound on geometric energy. In reality, this is ensured by the relation between geometric and topological energy.

3. Convergence of trajectories

Now we can use generalities on compactness aspects of J-holomorphic maps to see how to compactify the moduli spaces of Floer trajectories. We use the notation $\mathcal{M}(x^+, x^-, H_t, J)$ that we introduced before. Then we know that if $x^\pm$ is equipped with cappings $u^\pm : D^2 \to M$, we have

$$E^{geom}(u) = -\mathcal{A}([x^+, u^+]) + \mathcal{A}([x^-, u^-]) < \infty,$$

where the bound is uniform for all the elements in $\mathcal{M}(x^+, x^-, H_t, J)$.

For a Floer trajectory $u : \mathbb{R} \times S^1 \to M$, the punctures at infinities are not removable in general as u converges to Hamiltonian orbits. Nevertheless, to connect with our previous discussion, note that we have a conformal map

$$\mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}, \qquad (s, t) \mapsto e^{s+2\pi i t}.$$

The failure of sequential compactness can then be caused by blow-up of gradients near 0 or ∞, viewing $\mathbb{C} \setminus \{0\} \subset \mathbb{C}P^1$. Going back to the cylinder $\mathbb{R} \times S^1$, it means that the energy concentration can happen over the infinite parts of $\mathbb{R} \times S^1$. This is why you see *broken trajectories* in the compactification procedure. To emphasize the trajectory feature of Floer equations, we make the following simplifying assumption.

Assumption 3.1. (M, ω) *is symplectically aspherical, i.e.,* $\omega|_{\pi_2(M)} = 0$.

Definition 3.2. *An m-*translation vector $\underline{T}$ *is an m-tuple of real numbers*

$$T_1 < \cdots < T_m.$$

Theorem 3.3. *Given a sequence $u_k : \mathbb{R} \times S^1$ in $\mathcal{M}(x^+, x^-, H_t, J)$, after passing to a subsequence, we can find*

- *a collection of translation vectors $\underline{T}(k)$ with $T_j(k) - T_{j-1}(k) \xrightarrow{\alpha\to\infty} \infty$;*
- 1*-periodic orbts $x^- = x_0, x_1, \ldots, x_m = x^+$ and $v_j \in \mathcal{M}(x_j, x_{j+1}, H_t, J)$ for $j = 1, \ldots, m$;*

- *over any compact subsets of* $\mathbb{R} \times S^1$*, we have*

$$u_k(z + T_j(k)) \to v_j$$

uniformly in all derivatives; this convergence also holds in $W^{1,p}(\mathbb{R} \times S^1, M)$*;*
- *there is no loss of energy:*

$$\sum_j E^{geom}(v_j) = \lim_{k\to\infty} E(u_k).$$

Proof. Firstly, note that by applying the mean value inequality to the graph $\tilde{u}$ using the almost complex structure

$$\tilde{J} = \begin{pmatrix} j & 0 \\ J_t \circ X_{H_t} - X_{H_t} \circ j & J \end{pmatrix},$$

after conformally changing to $\mathbb{C} \setminus \{0\}$ to make the area of the cylinder factor sufficiently small, we see that there exists $\delta > 0$ such that for any sequence $v_k : [0, +\infty) \times S^1 \to M$ or $v_k : (-\infty, 0] \times S^1 \to M$ with $E^{geom}(v_k) \leq \delta$, we necessarily have $|dv_k| \leq C$ uniformly. Then using elliptic bootstrapping, we see that we can extract a subsequence of $\{v_k\}$ in this setting which converges uniformly in all derivatives over any compact subset to a limit solution to the Floer equation.

Let's choose ϵ to be the minimum of δ and the minimum of the energy gap $-\mathcal{A}(x^+) + \mathcal{A}(x^-)$ ranging over all pairs of distinct orbits $x^{\pm}$. Because $E^{geom}(u_k) > \epsilon$, we can find a unique $T_1(k)$ such that

$$\frac{1}{2}\int_{-\infty}^{T_1(k)} \int_{S^1} |du_k - X_{H_t}(u_k)| d\text{vol} = \epsilon.$$

Applying the translation by $T_1(k)$ in the domain of u_k, we may assume $T_1(k) = 0$. Then the new sequence $u_k(z + T_1(k))$ admits a limit v_1 modulo bubbling. Note that the bubbling cannot happen in the interior due to the assumption $\omega|_{\pi_2(M)} = 0$, otherwise the sphere bubble obtained by removal of singularity would have zero energy. In fact, there is no further bubbling at the negative ∞: this follows from our choice of ϵ. Therefore, we see that $u_k(z + T_1(k))$ convergences to v_1 uniformly in all derivatives over $(-\infty, 0] \times S^1$ and in all $W^{k,2}$.

If there is no bubbling at the positive ∞ in the convergence modulo bubbling $u_k^1 := u_k(z + T_1(k)) \rightharpoonup u_1$, we are already done. Otherwise, we see that

$$E^{geom}(u_k) - E^{geom}(v_1) > 0.$$

Then we can find $T_2(k) \in \mathbb{R}$ such that for the translated u_k^1, we have

$$\frac{1}{2}\int_{-\infty}^{T_2(k)} \int_{S^1} |du_k^1 - X_{H_t}(u_k^1)| d\text{vol} = \epsilon.$$

Then we can find another limit v_2 of the sequence $u_k^2 := u_k^1(z + T_2(k))$, modulo bubbling at the positive part. Note that the positive end of v_1 and the negative end of v_2 match with each other due to the exponential decay estimate.

We can then iterate this process. It must terminate because of the finiteness of $E^{geom}(u_k)$. This constructs the limit $(v_1, \ldots, v_m)$. The no-loss-of-energy follows from the construction. This and convergence modulo bubbling implies the uniform convergence over all compact subsets. □

Remark 3.4. (1) *We use the fact that any finite-energy solution to the Floer equation must converge to Hamiltonian orbits near infinity, and the limit is unique assuming non-degeneracy. See* [Sal99, Proposition 1.21].

(2) *If we drop the assumption* $\omega|_{\pi_2(M)} = 0$*, we have to deal with* stable Floer trajectories*, which will be discussed later.*

References

[MS04] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications, vol. 52, American Mathematical Society, 2004.

[Sal99] Dietmar Salamon, *Lectures on Floer homology*, Symplectic geometry and topology (Park City, UT, 1997), IAS/Park City Math. Ser., vol. 7, Amer. Math. Soc., Providence, RI, 1999, pp. 143–229.