# ABOUZAID–MCLEAN–SMITH

We will essentially follow Abouzaid–McLean–Smith's article *Complex cobordisms, Hamiltonian loops, and global Kuranishi charts* and focus on the geometric aspects of their main result.

## 1. Overview and motivations

Let us introduce some notations:

- $(X, \omega)$: closed symplectic manifold of real dimension $2n$,
- $\mathrm{Ham}(X, \omega)$: the group of Hamiltonian diffeomorphisms of $(X, \omega)$,
- $\mathrm{Symp}_0(X, \omega)$: the group of symplectomorphisms of $(X, \omega)$ isotopic to the identity,
- $\mathrm{Diff}_0(X)$: the group of diffeomorphisms of $X$ isotopic to the identity.

We have the inclusions

$$\mathrm{Ham}(X, \omega) \subseteq \mathrm{Symp}_0(X, \omega) \subseteq \mathrm{Diff}_0(X)$$

which are strict in general.

There is a classical *Flux* morphism

$$\mathrm{Flux} : \mathrm{Symp}_0(X, \omega) \longrightarrow H^1(X; \mathbb{R})$$

whose image $\Gamma$, the *flux group*, is known to be a discrete subgroup of $H^1(X; \mathbb{R})$ by the work of Ono. The discreteness of $\Gamma$ is equivalent to the fact that $\mathrm{Ham}(X, \omega)$ is dense in $\mathrm{Symp}_0(X, \omega)$ for the $C^1$ topology. In some sense, the flux group measures the difference between $\mathrm{Ham}(X, \omega)$ and $\mathrm{Symp}_0(X, \omega)$. Another way to measure this difference is to investigate the map

$$\pi_1 \mathrm{Ham}(X, \omega) \longrightarrow \pi_1 \mathrm{Symp}_0(X, \omega) \tag{1}$$

induced by the inclusion.

To each $[\phi] \in \pi_1 \mathrm{Symp}_0(X, \omega)$ one can associate a symplectic bundle

$$\begin{tikzcd} X \arrow[r, hook] & P_\phi \arrow[d] \\ & S^2 \end{tikzcd}$$

obtained by a clutching construction: glue two copies of $X \times D^2$ along $X \times S^1$ via the map

$$\begin{array}{ccc} S^1 \times X & \longrightarrow & S^1 \times X \\ (t, x) & \longmapsto & (-t, \phi_t(x)). \end{array}$$

Obviously, $P_\phi = X \times S^2$ if $[\phi] = [\mathrm{id}]$. To measure the complexity of a class $[\phi]$, one can measure the complexity of the bundle $P_\phi$. For instance, one can consider the (co)homology of $P_\phi$ with coefficients in $\mathbb{Q}$, $\mathbb{Z}$, or a more general ring (spectrum). For now, let us fix a ring $\Bbbk$. We will be interested in the maps

$$\iota : H_*(X; \Bbbk) \longrightarrow H_*(P_\phi; \Bbbk),$$
$$\rho : H^*(P_\phi; \Bbbk) \longrightarrow H^*(X; \Bbbk)$$

induced by the inclusion $X \hookrightarrow P_\phi$ as well as the *sweepout map*

$$\delta_\phi : H_*(X; \Bbbk) \to H_{*+1}(X; \Bbbk).$$

The latter is defined as the composition

$$H_*(X;\Bbbk) \xrightarrow{\cong} H_1(S^1;\Bbbk) \otimes_{\Bbbk} H_*(X;\Bbbk) \longrightarrow H_{*+1}(S^1 \times X;\Bbbk) \longrightarrow H_{*+1}(X;\Bbbk)$$

where the second map is induced by the Künneth morphism, and the last map is induced by $(t,x) \in S^1 \times X \longmapsto \phi_t(x) \in X$.

One may ask: when is the map $\delta_\phi$ trivial? Alternatively, does the cohomology of $P_\phi$ splits as $H^*(X;\Bbbk) \otimes_{\Bbbk} H^*(S^2;\Bbbk)$? For a general class $[\phi] \in \pi_1 \mathrm{Symp}_0(X,\omega)$, the answer is known to be negative. However, it holds for $[\phi] \in \pi_1 \mathrm{Ham}(X,\omega)$ and for suitable coefficient rings $\Bbbk$.

**Theorem 1** (Lalonde–McDuff–Polterovich for $(X,\omega)$ monotone, McDuff in general)**.** *Let $\Bbbk = \mathbb{Q}$ and $[\phi] \in \pi_1 \mathrm{Ham}(X,\omega)$. Then*

$$H^*(P_\phi;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \otimes H^*(S^2;\mathbb{Q})$$

*as $\mathbb{Q}$-vector spaces (we do not consider the ring structure on cohomology).*

Equivalently, still for $\Bbbk = \mathbb{Q}$,

- $\delta_\phi = 0$,
- $\iota : H_*(X;\mathbb{Q}) \to H_*(P_\phi;\mathbb{Q})$ is injective,
- The Serre spectral sequence of the fibration $X \hookrightarrow P_\phi \to S^2$ degenerates,
- $\rho : H^*(P_\phi;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is surjective.

Actually, the last four properties are always equivalent for any ring $\Bbbk$, and are implied by (but do not necessarily imply) the additive splitting $H^*(P_\phi;\Bbbk) \cong H^*(X;\Bbbk) \otimes H^*(S^2;\Bbbk)$. This result can be interpreted as an obstruction to the surjectivity of the map (1).

The main result of AMS is strengthening of Theorem 1 for $\Bbbk = \mathbb{Z}$ coefficients:

**Theorem 2** (Abouzaid–McLean–Smith)**.** *Let $\Bbbk = \mathbb{Z}$ and $[\phi] \in \pi_1 \mathrm{Ham}(X,\omega)$. Then*

$$H^*(P_\phi;\mathbb{Z}) \cong H^*(X;\mathbb{Z}) \otimes H^*(S^2;\mathbb{Z}) \tag{2}$$

*as $\mathbb{Z}$-modules. Therefore, the sweepout map $\delta_\phi : H_*(X;\mathbb{Z}) \to H_{*+1}(X;\mathbb{Z})$ vanishes.*

They actually prove an additive splitting for any *complex oriented* generalized cohomology theory.

From now on, we fix a class $[\phi] \in \pi_1 \mathrm{Ham}(X,\omega)$. To prove this result, we will show that $\rho$ is a *split-epimorphism* : there exists a $\mathbb{Z}$-module map

$$s : H^*(X;\mathbb{Z}) \longrightarrow H^*(P_\phi;\mathbb{Z})$$

which is a section of $\rho$, in the sense that $\rho \circ s = \mathrm{id}$. This is strictly stronger than $\rho$ being surjective in general.

**Lemma 3.** *Assume that $\rho$ is a split-epimorphism. Then* (2) *holds.*

*Proof.* We will omit the coefficients (assumed to be $\mathbb{Z}$) in the (co)homology groups, and we will use the notation $H^*(Y|A) = H^*(Y; Y \setminus A)$.

Notice that $P_\phi \setminus X$ is homotopy equivalent to $X$. We denote by $N \subset P_\phi$ a neighborhood of a fiber $X \subset P_\phi$ obtained as the preimage of a small disk under the fibration. Then by excision,

$$H^*(P_\phi|X) \cong H^*(N|X) \cong H^*(D^2 \times X|X) \cong H^*(D^2|\mathrm{pt}) \otimes H^*(X) \cong H^{*+2}(X).$$

The cohomology long exact sequence for the pair $(P_\phi, P_\phi \setminus X)$ splits as a short exact sequence

$$0 \longrightarrow H^*(P_\phi|X) \longrightarrow H^*(P_\phi) \longrightarrow H^*(P_\phi \setminus X) \longrightarrow 0$$

which is isomorphic to a short exact sequence

$$0 \longrightarrow H^{*+2}(X) \longrightarrow H^*(P_\phi) \xrightarrow{\rho} H^*(X) \longrightarrow 0,$$

and since $\rho$ admits a section, there is a isomorphism (depending on $s$)

$$H^*(P_\phi) \cong H^*(X) \oplus H^{*+2}(X) \cong H^*(S^2) \otimes H^*(X).$$

□

To construct $s$, we will consider suitable Gromov–Witten invariants in a larger symplectic fibration.

## 2. The geometric degeneration

Let $\mathbb{S}$ denote the one-point blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$. Composing the blowdown map $\mathbb{S} \to \mathbb{CP}^1 \times \mathbb{CP}^1$. with the projection onto the second factor yields a singular fibration $\pi_B : \mathbb{S} \to B = \mathbb{CP}^1$ which has one singular fiber over 0. This fiber is isomorphic to $\mathbb{CP}^1 \vee \mathbb{CP}^1$. For $t \in B$, we write $\mathbb{S}_t = \pi_B^{-1}(t)$.

It is not hard to see that the connected sum $P_\phi \#_X P_{\phi^{-1}}$ along a fiber is diffeomorphic to $P_{\mathrm{id}} = S^2 \times X$, and this space can be thought of as a resolution of the space $P_\phi \cup_X P_{\phi^{-1}}$. Therefore, there exists a smooth fibration $\pi_{\mathbb{S}} : \widetilde{P} \to \mathbb{S}$ such that $\pi_{\mathbb{S}}^{-1}(\mathbb{S}_0) \cong P_\phi \cup_X P_{\phi^{-1}}$, and $\pi_{\mathbb{S}}^{-1}(\mathbb{S}_t) \cong S^2 \times X$ for $t \neq 0$. Using that $\phi$ is a loop of *Hamiltonian* diffeomorphisms, McDuff upgraded this construction to a *symplectic* fibration:

**Proposition 4** (McDuff)**.** *There exists a* symplectic *fibration* $\pi_{\mathbb{S}} : \widetilde{P} \to \mathbb{S}$ *with fiber* $(X, \omega)$ *satisfying the following.*

1. *(Triviality at $\infty$) There exists a neighborhood $W_\infty \subset B$ of $\infty \in B$ over which $\pi_B$ is trivial, and such that the restriction of $\widetilde{P}$ to $\pi_B^{-1}(W_\infty)$ is isomorphic to the trivial symplectic fibration $(S^2 \times W_\infty \times X, \omega_{S^2} \oplus \omega_{S^2|W_\infty} \oplus \omega)$.*
2. *(Singular locus) $\pi_{\mathbb{S}}^{-1}(\mathbb{S}_0) \cong P_\phi \cup_X P_{\phi^{-1}}$, where each component is mapped to a reducible component of $\mathbb{S}_0 \cong \mathbb{CP}^1 \vee \mathbb{CP}^1$ and carries the canonical (deformation class of) symplectic structure.*

Let us introduce more notations:

- $\mathbb{S}_0 = \mathbb{S}_\phi \vee \mathbb{S}_{\phi^{-1}}$,
- $S^2_h$ is the image of a holomorphic section of $\pi_B : \mathbb{S} \to B$ passing through $\mathbb{S}_\phi \setminus \mathbb{S}_{\phi^{-1}}$,
- $(S^2 \times X)_h = \pi_{\mathbb{S}}^{-1}(S^2_h)$,
- For $t \in \mathbb{CP}^1 \setminus \{0\}$, $P_t = \widetilde{P}_{|\mathbb{S}_t}$.

Here, the subscript $h$ means “horizontal”. See Figure 1 for an illustration of the previous Proposition and some of these notations.

## 3. Moduli spaces of pseudoholomorphic curves

Let us choose a compatible almost complex structure $J$ on $\widetilde{P}$ which satisfies

- $\pi_{\mathbb{S}} : \widetilde{P} \to \mathbb{S}$ is pseudoholomorphic,
- $J$ is trivial over $S^2 \times W_\infty \times W$, i.e., it splits as a direct sum $J = j_{S^2} \oplus j_{S^2|W_\infty} \oplus J_X$, where $j$ denotes the canonical complex structure on $S^2$ and $J_X$ is a compatible almost complex structure on $X$.

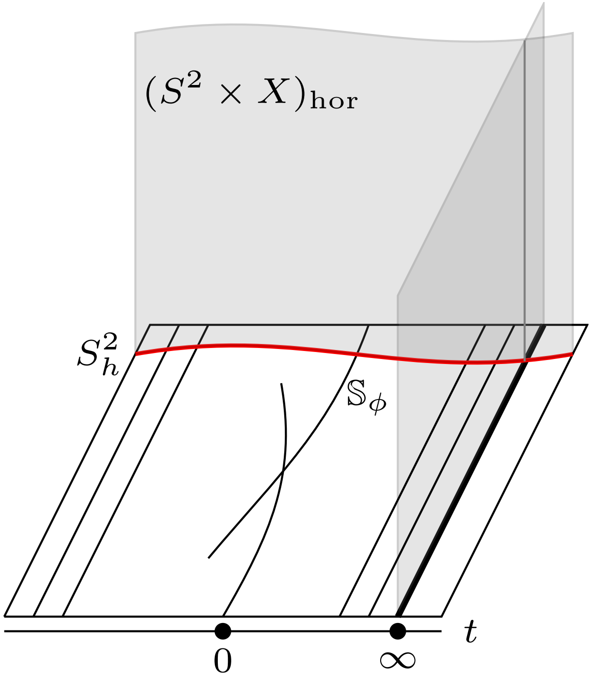

FIGURE 1. The fibration $\pi_{\mathbb{S}} : \widetilde{P} \to \mathbb{S}$ over $\pi_B : \mathbb{S} \to B$. Figure obtained from Bai–Xu's *An integral Euler cycle in normally complex orbifolds and* $\mathbb{Z}$*-valued Gromov–Witten type invariants* with the permission of the authors.

Let $A \in H_2(\widetilde{P};\mathbb{Z})$ denote the homology class represented by $S^2 \times \{\infty\} \times \{\mathrm{pt}\} \subset S^2 \times W_\infty \times X$. We denote by $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,2}(X, J; A)$ the moduli space of genus 0 stable $J$-holomorphic maps in the class $A$ with two marked points. It comes with two evaluation maps

$$\begin{tikzcd}\overline{\mathcal{M}} \arrow[r, shift left, "ev_1"] \arrow[r, shift right, "ev_2"'] & \widetilde{P}.\end{tikzcd}$$

We define

$$\overline{\mathcal{M}}_h := ev_1^{-1}((S^2 \times X)_h) \subset \overline{\mathcal{M}}$$

and for $\heartsuit \in \{\phi, \infty\}$,

$$\overline{\mathcal{M}}_\heartsuit := \overline{\mathcal{M}}_h \cap ev_2^{-1}(P_\heartsuit) \subset \overline{\mathcal{M}}_h.$$

Notice that because of or choice of $J$ and $A$, the curves in $\overline{\mathcal{M}}_\phi$ are contained in $P_\phi \cup_X P_{\phi^{-1}}$ (not necessarily in $P_\phi$!), and the curves in $\overline{\mathcal{M}}_\infty$ are contained in $P_\infty \cong S^2 \times X$. We obtain two correspondences

$$\begin{tikzcd} & \overline{\mathcal{M}}_\heartsuit \arrow[dl, "ev_1"'] \arrow[dr, "ev_2"] & \\ (S^2 \times X)_h & & P_\heartsuit \end{tikzcd}$$

which will induce two maps

$$H^*(S^2 \times X;\mathbb{Z}) \longrightarrow H^*(P_\heartsuit;\mathbb{Z}).$$

We will use the one for $\heartsuit = \phi$ to define the desired section $s$ of $\rho$, and we will compare it to the one for $\heartsuit = \infty$ to prove that $s$ is indeed a section of $\rho$. Let us first check that these moduli spaces have the correct virtual dimensions.

**Lemma 5.** *$\overline{\mathcal{M}}_h$ has virtual dimension $\dim(\widetilde{P}) = 2n+4$, and $\overline{\mathcal{M}}_\heartsuit$ has virtual dimension $\dim(P_\heartsuit) = 2n+2$.*

*Proof.* First, note that $\langle c_1(T\widetilde{P}), A\rangle = \chi(S^2) = 2$. Hence, the virtual dimension of the moduli space of *parametrized* genus 0 stable curves in $\widetilde{P}$ in the class $A$ without marked points is $\dim(\widetilde{P}) + 2\langle c_1(T\widetilde{P}), A\rangle = 2n+8$. We can assume that the added marked points lie are two prescribed points, e.g., at 0 and $\infty$, leaving a $\mathbb{C}^*$ reparametrization action which drops the virtual dimension by 2, hence $\mathrm{vdim}\overline{\mathcal{M}} = 2n+6$. Adding a divisorial constraint on one marked point drops the virtual dimension by 2, hence $\mathrm{vdim}\overline{\mathcal{M}}_h = 2n+4$, and adding another divisorial constraint on the other marked point yields $\mathrm{vdim}\overline{\mathcal{M}}_\heartsuit = 2n+2$. ☐

Of course, these moduli spaces might not have a natural structure of smooth manifolds of the correct dimension. Let us make the following

**Assumption.** *(Transversality) $\overline{\mathcal{M}}_h$ is a smooth oriented manifold of dimension $2n+4$, and the evaluation map $ev_2 : \overline{\mathcal{M}}_h \to \widetilde{P}$ is a smooth submersion transverse to $P_\infty$ and $P_\phi$, so that $\overline{\mathcal{M}}_\heartsuit$, $\heartsuit \in \{\infty, \phi\}$, are smooth manifolds of dimension $2n+2$.*

Under this (unrealistic!) transversality assumption, we explain how to prove Theorem 2. For $\heartsuit \in \{\infty, \phi\}$, we define a map $s_\heartsuit : H^*(X;\mathbb{Z}) \longrightarrow H^*(P_\heartsuit;\mathbb{Z})$ as the composition

$$\begin{tikzcd}
& H^*(\overline{\mathcal{M}}_\heartsuit) \arrow[r, "\cap[\overline{\mathcal{M}}_\heartsuit]"] & H_{2n+2-*}(\overline{\mathcal{M}}_\heartsuit) \arrow[dr, "(ev_2)_*"] & \\
H^*((S^2 \times X)_h) \arrow[ur, "ev_1^*"] & & & H_{2n+2-*}(P_\heartsuit) \arrow[d, "\mathrm{PD}"] \\
H^*(X) \arrow[u, "\mathrm{pr}_2^*"] \arrow[rrr, "s_\heartsuit"'] & & & H^*(P_\heartsuit)
\end{tikzcd}$$

where we dropped the coefficients from the notations. We now show that $s = s_\phi$ is the desired section of $\rho : H^*(P_\phi;\mathbb{Z}) \to H^*(X;\mathbb{Z})$.

**Lemma 6.** *$\rho \circ s_\phi = \mathrm{id}$.*

*Proof.* We consider the following commutative diagram:

$$\begin{tikzcd}
& H^*(\overline{\mathcal{M}}_h) \arrow[r, "\mathrm{PD}"] \arrow[dd, "inc^*"] & H_{2n+4-*}(\overline{\mathcal{M}}_h) \arrow[r, "(ev_2)_*"] \arrow[dd, "inc^!"] & H_{2n+4-*}(\widetilde{P}) \arrow[r, "\mathrm{PD}"] \arrow[dd, "inc^!"] & H^*(\widetilde{P}) \arrow[dd, "inc^!"] \arrow[dr] & \\
H^*((S^2 \times X)_h) \arrow[ur, "ev_1^*"] \arrow[dr, "ev_1^*"'] & & & & & H^*(X) \\
H^*(X) \arrow[u, "pr_2^*"] \arrow[rrrr, bend right, "s_\heartsuit"'] & H^*(\overline{\mathcal{M}}_\heartsuit) \arrow[r, "\mathrm{PD}"] & H_{2n+2-*}(\overline{\mathcal{M}}_\heartsuit) \arrow[r, "(ev_2)_*"] & H_{2n+2-*}(P_\heartsuit) \arrow[r, "\mathrm{PD}"] & H^*(P_\heartsuit) \arrow[ur, "\rho_\heartsuit"'] &
\end{tikzcd}$$

This implies that $\rho \circ s = \rho_\phi \circ s_\phi = \rho_\infty \circ s_\infty$. However, the map $s_\infty$ is easy to compute since the curves in $\overline{\mathcal{M}}_\infty$ can be described explicitly. In fact, any curve in $\overline{\mathcal{M}}_\infty$ is of the form

$$\begin{array}{rccc} u: & S^2 & \longrightarrow & P_\infty = S^2 \times \{\infty\} \times X \\ & z & \longmapsto & (\psi(z), \infty, \mathrm{pt}), \end{array}$$

where $\psi : S^2 \to S^2$ is a biholomorphism. Therefore, $\overline{\mathcal{M}}_\infty \cong P_\infty = S^2 \times \{\infty\} \times X$ and the evaluation map $ev_2 : \overline{\mathcal{M}}_\infty \to P_\infty$ is a diffeomorphism, which implies that $s_\infty = pr_2^*$ and $\rho_\infty \circ s_\infty = \mathrm{id}$, as desired. ☐

## 4. Removing the transversality assumption

We know explain how to make the previous argument work without the transversality assumption on the moduli spaces. There are (at least) two approaches.

- One strategy consists of defining a $\mathbb{Z}$-valued fundamental class for the Gromov–Witten moduli spaces using suitable global Kuranishi charts/derived orbifold charts. This is carried out in Bai–Xu's *An integral Euler cycle in normally complex orbifolds and $\mathbb{Z}$-valued Gromov–Witten type invariants*, using their *FOP perturbations*.
- Another perhaps less direct strategy is to work over *Morava $K$-theory* to show that $\rho$ is a split-epimorphism for $\mathbb{Z}/p^k\mathbb{Z}$ coefficients, for every prime $p$ and every integer $k \geq 1$. This is the approach of Abouzaid–McLean–Smith. To that extent, they construct a *global Kuranishi chart* for the moduli spaces of interest, which satisfy suitable orientation properties with respect to Morava $K$-theory.

We will focus on the AMS approach and briefly overview the main ingredients. Let us first discuss Morava $K$-theory.

For a prime $p$ and an integer $n \geq 1$, $\mathbb{k} = K_p(n)$ is a generalized cohomology theory which satisfies the following:

1. The underlying coefficient ring is $\mathbb{k}_* = H^*(\mathrm{pt}, \mathbb{k}) = \mathbb{F}_p[v^\pm]$ where $|v^\pm| = 2(p^n - 1)$,
2. If $X$ and $Y$ are CW-complexes, there is a Künneth isomorphism
$$H^*(X \times Y; \mathbb{k}) \cong H^*(X; \mathbb{k}) \otimes_{\mathbb{k}_*} H^*(Y; \mathbb{k}),$$
3. Every vector bundle with a stable complex structure is $\mathbb{k}$-oriented, hence every stably complex manifold is $\mathbb{k}$-oriented,
4. If $p > 2$, then any oriented vector bundle is $\mathbb{k}$-oriented, hence every oriented manifold is $\mathbb{k}$-oriented.

For $k \geq 1$, there is an extension of $K_p(n)$ denoted $K_{p^k}(n)$ with coefficient ring $\mathbb{Z}/p^k\mathbb{Z}(n)[v^\pm]$, $|v| = 2(p^n - 1)$, which satisfies (3) and (4).

The upshot of AMS's proof is that $K_{p^k}(n)$ behaves a bit like a coefficient field, and their global Kuranishi charts are $K_{p^k}(n)$-oriented, allowing them to construct virtual fundamental classes with coefficients in $K_{p^k}(n)$. One crucial aspect is that Morava $K$-theories satisfy a version of equivariant Poincaré duality, or rather Atiyah duality, which is used to define suitable pushforward maps in cohomology (those are relevant for the definition of $s$ as in the previous section). This equivariant duality statement was proved by Cheng.

The virtual fundamental class then takes the form of a map $H^*(M; \mathbb{k}) \to \mathbb{k}_*$. More generally, for a space $M$ with a suitable global Kuranishi chart $\mathcal{K} = (G, \mathcal{T}, E, s)$ and a map $f : M \to X$ to a $\mathbb{k}$-oriented smooth manifold, AMS construct a pushforward map

$$f_*^{\mathcal{K}} : H^*(M; \mathbb{k}) \longrightarrow H^{*-\mathrm{vdim}(M)+\dim(X)}(X; \mathbb{k})$$

which behaves naturally with respect to restricting to subspaces of the form $f^{-1}(S)$ for a $\Bbbk$-oriented submanifold $S \subseteq X$ (for a suitable induced Kuranishi chart on $f^{-1}(S)$). Therefore, one can make sense of the diagrams in the previous section for cohomology groups with $K_{p^k}(n)$-coefficients. This implies:

**Lemma 7.** *The map $\rho : H^*(P_\phi; K_{p^k}(n)) \to H^*(X; K_{p^k}(n))$ is a split-epimorphism.*

Now since $P_\phi$ and $X$ are finite dimensional manifolds, for $n$ large enough $(2(p^n-1) > 2n+4)$, the Atiyah-Hirzebruch spectral sequence for $K_{p^k}(n)$ degenerates and

$$H^*(X; K_{p^k}(n)) \cong H^*(X; \mathbb{Z}/p^k\mathbb{Z}) \otimes_{\mathbb{Z}/p^k\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}[v^{\pm}],$$
$$H^*(P_\phi; K_{p^k}(n)) \cong H^*(P_\phi; \mathbb{Z}/p^k\mathbb{Z}) \otimes_{\mathbb{Z}/p^k\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}[v^{\pm}],$$

which implies that $H^*(P_\phi; \mathbb{Z}/p^k\mathbb{Z}) \to H^*(X; \mathbb{Z}/p^k\mathbb{Z})$ is a split-epimorphism. Since this holds for every prime $p$ and every integer $k$, $\rho : H^*(P_\phi; \mathbb{Z}/m\mathbb{Z}) \to H^*(X; \mathbb{Z}/m\mathbb{Z})$ is also a split-epimorphism for every integer $m \geq 1$, which implies that $\rho : H^*(P_\phi; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ is a split-epimorphism.