Topic 5: Pontryagin–Thom construction and Kuranishi reduction

1. Pontryagin–Thom

Two closed smooth n-manifolds M and N are said to be (unoriented) bordant to each other if there exists an $(n+1)$-dimensional manifold-with-boundary W such that $\partial = M \coprod N$.

Definition 1.1. *The unoriented bordism group $\mathcal{N}_*$ is the graded abelian group whose degree n part $\mathcal{N}_n$ is the set of isomorphism classes of n-dimensional smooth closed manifolds modulo the bordism relation, where the addition law is given by disjoint union.*

As $M \coprod M$ is the boundary of $M \times [0,1]$, every element in $\mathcal{N}_*$ is 2-torsion and the inverse of $[M]$ is itself. This is also a ring, with product structure induced from the Cartesian product.

Questions 1.2. *How to calculate $\mathcal{N}_*$?*

We use the following procedure, the so-called *Pontryagin–Thom construction*, to turn $\mathcal{N}_*$ into a homotopical object. Given a smooth closed n-dimensional manifold M, using the Whitney embedding theorem, we can find a smooth embedding

$$i : M \hookrightarrow \mathbb{R}^{n+k}$$

for some $k > 0$. Then we have the following short exact sequence of vector bundles

$$0 \longrightarrow TM \longrightarrow i^*\underline{\mathbb{R}}^{n+k} \longrightarrow \nu \longrightarrow 0,$$

where $\nu \to M$ is the normal bundle of M under the embedding i. We can use the exponential map with respect to the Euclidean metric to define an open embedding from the unit disk-bundle

$$D(\nu) \hookrightarrow \mathbb{R}^{n+k},$$

which can be equivalently thought of as the tubular neighborhood.

For any two choices

$$i : M \hookrightarrow \mathbb{R}^{n+k}, \qquad i' : M \hookrightarrow \mathbb{R}^{n+k'},$$

with the corresponding open embeddings

$$D(\nu) \hookrightarrow \mathbb{R}^{n+k}, \qquad D(\nu') \hookrightarrow \mathbb{R}^{n+k'},$$

they can be related in the following way, which can be proved using the arguments of the Whitney embedding theorem.

Theorem 1.3. *There exists $K \geq \max\{k, k'\}$ such that for the embeddings $M \xrightarrow{i} \mathbb{R}^{n+k} \to \mathbb{R}^{n+K}$ and $M \xrightarrow{i'} \mathbb{R}^{n+k'} \to \mathbb{R}^{n+K}$, the induced open embedding from the tubular neighborhoods are isotopic to each other.*

In particular, the vector bundles ν and ν' are stably isomorphic:

$$\nu \oplus \underline{\mathbb{R}}^{K-k} \cong \nu' \oplus \underline{\mathbb{R}}^{K-k'}.$$

We call this stable isomorphism class of vector bundles over M its *stable normal bundle.*

Given a vector bundle $\xi \to M$, recall that its *Thom space* $\mathrm{Th}(\xi)$ is defined to be

$$\mathrm{Th}(\xi) := D(\xi)/S(\xi),$$

where we abuse the notation to use $D(\xi)$ denote the closed unit disk-bundle of ξ under a given metric, and $S(\xi)$ is the unit sphere-bundle. $\mathrm{Th}(\xi)$ is a pointed-space with base point at infinity. Going back to $i : M \hookrightarrow \mathbb{R}^{n+k}$, we can define a map

$$S^{n+k} = \mathbb{R}^{n+k} \coprod \{\infty\} \to \mathrm{Th}(\nu)$$

which takes the complement of $D(\nu) \subset \mathbb{R}^{n+k}$ to the base point, and is the identity map over $D(\nu)$. This is called the *Pontryagin–Thom collapsing map.*

Exercise 1.4. *Show that for $m > 0$, we have*

$$\mathrm{Th}(\xi \oplus \underline{\mathbb{R}}^m) = S^m \wedge \mathrm{Th}(\xi) = \Sigma^m(\mathrm{Th}(\xi)),$$

where Σ is the (reduced) suspension on based topological spaces.

Accordingly, for the composed embedding $M \xrightarrow{i} \mathbb{R}^{n+k} \to \mathbb{R}^{n+K}$, the induced map

$$S^{n+K} \to \mathrm{Th}(\nu \oplus \underline{\mathbb{R}}^{K-k})$$

comes from suspending the original map $S^{n+k} \to \mathrm{Th}(\nu)$ by $K - k$ times. Then by Theorem 1.3, for two different choices of embeddings, we see that

$$S^{n+K} \to \mathrm{Th}(\nu \oplus \underline{\mathbb{R}}^{K-k}), \qquad S^{n+K} \to \mathrm{Th}(\nu' \oplus \underline{\mathbb{R}}^{K-k'})$$

define the same homotopy class of maps.

Now let's take a closer look at ν. Given a point in ν, it gives rise to

(1) an k-dimensional subspace of $\mathbb{R}^{n+k}$, the normal space, which is an element in $\mathrm{Gr}(k, n+k)$, the Grassmannian of k-planes in $\mathbb{R}^{n+k}$;
(2) a vector in this k-dimensional vector space, the normal vector itself, so an element of the tautological vector bundle $\xi_k \to \mathrm{Gr}(k, n+k)$.

As a result, we obtain a map

$$\mathrm{Th}(\nu) \to \mathrm{Th}(\xi_k),$$

which can be composed with the PT collapsing map to give

$$S^{n+k} \to \mathrm{Th}(\xi_k).$$

By the independence result for different choices of embeddings, we see that M defines a unique class in

$$\lim_{k\to\infty} [S^{n+k}, \mathrm{Th}(\xi_k)].$$

Furthermore, we can replace $\xi_k \to \mathrm{Gr}(k, n+k)$ by the bundle $\xi_k \to BO(k)$, and the Thom space $\mathrm{Th}(\xi_k)$ over $BO(k)$ is usually written as $MO(k)$.

Theorem 1.5. *The oriented bordism group* $\mathcal{N}_n$ *is isomorphic to*

$$\lim_{k\to\infty} \pi_{n+k}(MO(k)).$$

Proof. We have described the map

$$\mathcal{N}_n \to \lim_{k\to\infty} \pi_{n+k}(MO(k))$$

in the above discussions. To show it's well-defined, we need to argue that it descends under the bordism relation. To this end, for a manifold-with-boundary W inducing the bordism relation of M and N, we can choose an embedding

$$(W, M \coprod N) \to (\mathbb{R}^{n+k} \times [0,1], \mathbb{R}^{n+k} \times \{0\}, \mathbb{R}^{n+k} \times \{1\})$$

using the collar coordinates and a stabilization. By performing the Pontryagin–Thom collapsing construction over $[0,1]$, we see that the map indeed descends.

There is also something interesting in the reversed direction. Representing an element in the stable homotopy group $\lim_{k\to\infty} \pi_{n+k}(MO(k))$ by a map

$$S^{n+k} \to \pi_{n+k}(MO(k)),$$

we need to construct a manifold. Given the diagram

$$\begin{array}{ccc} & \nearrow & \mathrm{Th}(\xi_k) \\ & & \downarrow \\ S^{n+k} & \longrightarrow & BO(k), \end{array}$$

it is equivalent to

$$\begin{array}{ccc} & \overset{\tilde{f}}{\nearrow} & \xi_k \\ & & \downarrow \\ \mathbb{R}^{n+k} & \underset{f}{\longrightarrow} & BO(k) \end{array}$$

where the map $\tilde{f}$ is proper. This datum determines a vector bundle $f^*\xi_k \to \mathbb{R}^{n+k}$ together with a section $s : \mathbb{R}^{n+k} \to f^*\xi_k$ determined by the lift $\tilde{f}$. By the *Thom transversality theorem*, we can choose the representative such that $\tilde{f}$ is transverse to the zero section $BO(k) \xrightarrow{0} \xi_k$, which in turn implies that s is transverse to the zero section of $f^*\xi_k \to \mathbb{R}^{n+k}$. For such a choice, $s^{-1}(0)$ is a closed smooth manifold. Moreover, for two difference choices of transverse $\tilde{f}$, we can find a homotopy between them which remains transverse to $BO(k) \xrightarrow{0} \xi_k$. Then the resulting sections of $f^*\xi_k \to \mathbb{R}^{n+k}$ cut out bordant closed manifolds. This defines the map

$$\lim_{k\to\infty} \pi_{n+k}(MO(k)) \to \mathcal{N}_n,$$

whose independence on the choice of representatives $S^{n+k} \to \pi_{n+k}(MO(k))$ can be shown by a stabilization argument.

One can then see that these two maps are inverse to each other. For one direction, the tautological section of $\pi_\nu^*\nu \to \nu$ for $\pi_\nu : \nu \to M$ indeed cuts out M; for the other direction, the Pontragin–Thom collapsing map indeed reconstructs the original map from the sphere. □

The object $\lim_{k\to\infty} \pi_{n+k}(MO(k))$ is easier to manipulate with, and the homotopy groups are eventually calculated by the homology of $BO(k)$. One important consequence is the following theorem by Thom.

Theorem 1.6. *For any manifold X and a homology class $\alpha \in H_k(X, \mathbb{F}_2)$, there exists a closed manifold M of dimension k such that*

$$f_*([M]) = \alpha,$$

where $[M] \in H_k(M, \mathbb{F}_2)$ is the fundamental class and $f : M \to X$ is a continuous map.

2. Equivariant situation

Let Γ be a finite group. We can then define the Γ-equivariant (unoriented) bordism group

$$\mathcal{N}_*^\Gamma$$

consisting of isomorphism classes of smooth closed Γ-manifolds modulo the Γ-equivariant bordism relation, where the addition law is again given by disjoint union. It also has a ring structure, where Γ acts on $M \times N$ diagonally.

We can consider the analog of $\xi_k \to BO(k)$, replacing $BO(k)$ by the classifying space of Γ-equivariant vector bundles $BO(k)_\Gamma$, over which lives the tautological bundle

$$\xi_k^\Gamma \to BO(k)_\Gamma.$$

By restricting to connected components, we can prescribe the isomorphism type of the representation of the fiber. We will denote the tautological bundle over the connected component of $BO(k)_\Gamma$ indexed by the Γ-representation V by

$$\xi_V \to BO(V)_\Gamma.$$

Then we can consider

$$\lim_V [S^{V\oplus\mathbb{R}^n}, \mathrm{Th}(\xi_V)]^\Gamma,$$

(1) here we consider the direct system of finite-dimensional Γ representations;
(2) $S^{V\oplus\mathbb{R}^k}$ is the one-point compactification of the Γ-representation $V \oplus \mathbb{R}^k$ where Γ acts trivially on $\mathbb{R}^k$;
(3) we only take homotopy classes of Γ-equivariant maps.

Exercise 2.1. *Show that there is a well-defined map*

$$\mathcal{N}_n^\Gamma \to \lim_V [S^{V\oplus\mathbb{R}^n}, \mathrm{Th}(\xi_V)]^\Gamma.$$

The important insight here is that there is an equivariant analog of the Whitney embedding theorem: for a compact Γ-manifold M, there exists a finite-dimensional Γ-representation V such that there exists a Γ-equivariant embedding $M \hookrightarrow V$. If we allow V to be infinite-dimensional, then we can simply take V to be the space of L^2 functions over M with the pull-back Γ-action. To go back to finite dimensions, the point is that a sufficiently large approximation of $L^2(M)$ is good enough to separate points and tangent vectors.

Now let's investigate the other direction of the Pontryagin–Thom construction. We shall look at

$$\begin{array}{ccc} & & \xi_V \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ V \oplus \mathbb{R}^n & \xrightarrow[f]{} & BO(V)_\Gamma. \end{array}$$

To get a Γ-manifold, we have to choose a representative $\tilde{f}$ which is Γ-equivariant and transverse to $BO(V)_\Gamma \xrightarrow{0} \xi_V$. However, this *cannot* be achieved in general. As a typical example, consider $\mathbb{R}$ with two different $\mathbb{Z}/2$-actions: for the first copy $\mathbb{R}(0)$, the groups acts on it trivially; for the second copy $\mathbb{R}(-1)$, we treat it as the sign representation. Then any $\mathbb{Z}/2$-equivariant map $\mathbb{R}(0) \to \mathbb{R}(-1)$ is necessarily the map which takes all points to 0. In other words, we can never find a section of the vector bundle $\underline{\mathbb{R}(-1)} \to \mathbb{R}(0)$ which is transverse to the zero section.

Nevertheless, we can consider the Γ-equivariant vector bundle

$$\tilde{f}^* \xi_V \to V \oplus \mathbb{R}^n,$$

which comes with a proper section $s : V \oplus \mathbb{R}^n \to \tilde{f}^* \xi_V$ induced from $\tilde{f}$. This triple

$$(V \oplus \mathbb{R}^n, \tilde{f}^* \xi_V, s)$$

records enough homotopical information to remedy the failure of the Pontryagin–Thom isomorphism in the equivariant setting.

Definition 2.2. *A* Γ-equivariant Kuranishi chart *is a triple*

$$(M, E, s)$$

where M is a Γ-manifold, $E \to M$ is a Γ-equivariant vector bundle, $s : M \to E$ is a Γ-equivariant section such that $s^{-1}(0)$ is compact.

Motivated by the desire to salvage the Pontryagin–Thom isomorphism, what we really care about a Γ-equivariant Kuranishi chart (M, E, s) is actually behavior of $s^{-1}(0)$ and how it can be thickened up to be a smooth manifold. Therefore, we consider the following relations:

(1) Shrinking: it replaces (M, E, s) by $(M', E|_{M'}, s|_{M'})$, where $M' \subset M$ is an open neighborhood of $s^{-1}(0)$.
(2) Stabilization: given another Γ-equivariant vector bundle $\pi_F : F \to M$, it replaces (M, E, s) by
$$(F, \pi_F^* F \oplus \pi_F^* E, \tau_F \oplus \pi^* s)$$
where $\tau : F \to \pi_F^* F$ is the tautological section.

The notion of bordism can be extended to triples: we say (M, E, s) and (M', E', s') are bordant if there exists a Γ-manifold with boundary W, and a Γ-equivariant vector bundle $\tilde{E} \to W$ with a Γ-equivariant section $S : W \to \tilde{E}$ such that

$$(\partial W, \tilde{E}|_{\partial W}, S|_{\partial W}) = (M \coprod M', E \coprod E', s \coprod s').$$

Then we can define a new bordism group

$$d\mathcal{N}_*$$

which consists of isomorphism classes of Γ-equivariant Kuranishi charts, modulo the equivalence reltions generated by shrinking, stabilization, and bordism. It is an abelian group under disjoint union.

Proposition 2.3. *There exists an isomorphism*

$$d\mathcal{N}_n \to \lim_{V} \, [S^{V \oplus \mathbb{R}^n}, \mathrm{Th}(\xi_V)]^\Gamma$$

from the Pontryagin–Thom construction.

The only missing piece is the following: given (M, E, s), the bundle E admits a stable inverse in the sense that we can find a Γ-equivariant vector bundle $\pi_F : F \to M$ such that $E \oplus F \cong \underline{V}$ for some Γ-representation V. Using the stabilization relation, we can replace (M, E, s) by $(F, \underline{V}, \pi_F^* s \oplus \tau_F)$. We need to find a commutative square

$$\begin{array}{ccc} \underline{V} & \longrightarrow & \xi_V \\ {\scriptstyle \pi_F^* s \oplus \tau_F} \uparrow\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow \\ F & \longrightarrow & BO(V)_\Gamma \end{array}$$

where $\tilde{f} : F \to \xi_V$ is induced by a Whitney embedding into a representation. This can be done even if F is not compact: it only has finitely many orbit types.

Exercise 2.4. *Complete the above construction and prove the Pontryagin–Thom isomorphism in the equivariant setting.*

The group $d\mathcal{N}_*$ is usually called the *homotopical* Γ-equivariant bordism group.

3. Kuranishi reduction

What's the relation between the above classical algebraic topology and moduli spaces of pseudo-holomorphic curves? The answer is, via a finite-dimensional reduction, called the *Kuranishi reduction*, Kuranishi charts naturally arise from nonlinear Fredholm problems.

Let $\mathcal{X}, \mathcal{Y}$ be (possibly finite dimensional) Banach spaces and let

$$\Phi : \mathcal{X} \to \mathcal{Y} \tag{3.1}$$

be a smooth Fredholm map, defined on an open neighborhood of $0 \in \mathcal{X}$, such that we have $\Phi(0) = 0 \in \mathcal{Y}$. Consider the linearization

$$\tilde{D} = D\Phi(0) : \mathcal{X} \to \mathcal{Y} \tag{3.2}$$

and define the (necessarily finite dimensional) vector spaces $T := \ker \tilde{D}$ and $E := \operatorname{coker}\tilde{D}$. Split the inclusion $T \to \mathcal{X}$ and the projection $\mathcal{Y} \to E$ by choosing a Banach space complement $\mathcal{X}' \subset \mathcal{X}$ of T and a linear subspace lift $E \subset \mathcal{Y}$. Applying the implicit function theorem now shows that $\Phi^{-1}(E) \subset \mathcal{X}$ is identified (near $0 \in \mathcal{X}$) with the graph of a smooth map $\Psi : T \to \mathcal{X}'$ defined on a neighborhood of $0 \in T$. This map satisfies $\Psi(0) = 0 \in \mathcal{X}'$ and $D\Psi(0) = 0$. The associated *Kuranishi map*

$$F : T \to E, \tag{3.3}$$

defined near $0 \in T$, is given by $F(\kappa) := \Phi(\kappa + \Psi(\kappa))$ for $\kappa \in T$. It satisfies $F(0) = 0$ and $DF(0) = 0$ by construction. We refer to the process of constructing F from Φ as *Kuranishi reduction* and note that it depends on the choices of splittings of the maps $T \to \mathcal{X}$ and $\mathcal{Y} \to E$.

Lemma 3.1 (Splitting dependence of Kuranishi map)**.** *Let $\tilde{F} : T \to E$ be a Kuranishi map obtained using a different choice of splittings than in* (3.3)*. Then, there exists a local diffeomorphism*

$$\psi : T \to T \tag{3.4}$$

with $\psi(0) = 0$ and $D\psi(0) = \mathrm{id}_T$ and a smooth map

$$\tilde{\psi} : T \to \mathrm{End}(E) \tag{3.5}$$

with $\tilde{\psi}(0) = \mathrm{id}_E$ such that, for all $\kappa \in T$ near 0, we have

$$\tilde{F}(\kappa) = \tilde{\psi}(\kappa) F(\psi(\kappa)). \tag{3.6}$$

Proof. Via the splitting used to define (3.3), we get direct sum decompositions $\mathcal{X} = \mathcal{X}' \oplus T$ and $\mathcal{Y} = \mathrm{im}(\tilde{D}) \oplus E$.

Consider first the case when $\tilde{F}$ is obtained by the changing the splitting of $T \to \mathcal{X}$ but using the same splitting of $\mathcal{Y} \to E$ as in (3.3). Denote the resulting complement of T by $\tilde{\mathcal{X}}' \subset \mathcal{X}$. Using the implicit function theorem, we write $\Phi^{-1}(E)$ as the graph of a smooth map $\tilde{\Psi} : T \to \tilde{\mathcal{X}}'$ near $0 \in \mathcal{X}$, with $\tilde{\Psi}(0) = 0$ and $D\tilde{\Psi}(0) = 0$. Since $\Phi^{-1}(E)$ is locally also the graph of the smooth map $\Psi : T \to \mathcal{X}'$, we must have

$$\Pi_{\mathcal{X}'} \tilde{\Psi}(\kappa) = \Psi(\kappa + \Pi_T \tilde{\Psi}(\kappa)) \tag{3.7}$$

for all $\kappa \in T$ near 0, where $\Pi_{\mathcal{X}'}$ and Π_T denote the projections onto the summands of the decomposition $\mathcal{X} = \mathcal{X}' \oplus T$. This immediately yields $\tilde{F}(\kappa) \equiv F(\psi(\kappa))$ with $\psi : T \to T$ defined by

$$\psi(\kappa) = \kappa + \Pi_T \tilde{\Psi}(\kappa). \tag{3.8}$$

Since $\tilde{\Psi}(0) = 0$ and $D\tilde{\Psi}(0) = 0$, we see that ψ satisfies $\psi(0) = 0$ and $D\psi(0) = \mathrm{id}_T$ and is thus a local diffeomorphism.

Consider next the case when $\tilde{F}$ is obtained by keeping the same splitting of $T \to \mathcal{X}$ as in (3.3) but changing the splitting of the map $\mathcal{Y} \to E$. Denote the resulting complement $\mathrm{im}(\tilde{D})$ by $\tilde{E} \subset \mathcal{Y}$. Consider the map

$$\Phi_E : \mathcal{X} \oplus E \to \mathcal{Y} \tag{3.9}$$

given by $\Phi_E(x,e) = \Phi(x) - e$. Using the implicit function theorem, we may write $\Phi_E^{-1}(\tilde{E})$, near 0, as the graph of a smooth map $\tilde{\Psi}_E : T \oplus E \to \mathcal{X}'$. Using $\tilde{\Psi}_E$, we define the smooth map $\tilde{F}_E : T \oplus E \to E$ by the formula

$$\tilde{F}_E(\kappa, e) = \Pi_E(\Phi(\kappa + \tilde{\Psi}_E(\kappa, e)) - e) \tag{3.10}$$

where $\Pi_E : \mathcal{Y} \to E$ denotes the projection. Restricting $\tilde{\Psi}_E$ to $T \oplus \{0\}$ recovers the map $\tilde{\Psi} : T \to \mathcal{X}'$ whose graph coincides with $\Phi^{-1}(\tilde{E})$ near $0 \in \mathcal{X}$ (which also shows that $\tilde{F}_E(\kappa, 0) = \tilde{F}(\kappa)$ for $\kappa \in T$). Therefore, as before, we have $\tilde{\Psi}(0) = 0$ and $D\tilde{\Psi}(0) = 0$ which shows that $D\tilde{F}_E(0,0)$ is given by (the negative of) the projection onto E. Now, if (κ, e) is a point where $\tilde{F}_E$ vanishes, then since $\Pi_E|_{\tilde{E}} : \tilde{E} \to E$ is an isomorphism it follows that $\Phi(\kappa + \tilde{\Psi}_E(\kappa, e)) - e = 0$. This, in turn, implies that we must in fact have $\tilde{\Psi}_E(\kappa, e) = \Psi(\kappa)$ and $e = F(\kappa)$. Conversely, if $e = F(\kappa)$, then $\Phi_E(\kappa + \Psi(\kappa), e) = \Phi(\kappa + \Psi(\kappa)) - e = 0 \in \tilde{E}$ and thus, $\tilde{\Psi}_E(\kappa, e) = \Psi(\kappa)$ and $\tilde{F}_E(\kappa, e) = 0$. In summary, $\tilde{F}_E$ is a defining equation, near $(0,0)$, for the submanifold of $T \oplus E$ given by the graph of the map $F : T \to E$, i.e., the equation $\tilde{F}_E = 0$ transversely cuts out the graph of F near $(0,0)$. Since $F_E(\kappa, e) := F(\kappa) - e$ is also a defining equation for the graph of F, and $DF_E(0,0) = D\tilde{F}_E(0,0)$, it follows that there exists a smooth map $\tilde{\psi}_E : T \oplus E \to \mathrm{End}(E)$ with $\tilde{\psi}_E(0,0) = \mathrm{id}_E$ such that

$$\tilde{F}_E(\kappa, e) = \tilde{\psi}_E(\kappa, e) \cdot F_E(\kappa, e) \tag{3.11}$$

for all $(\kappa, e) \in T \oplus E$ near $(0,0)$. Restricting to $e = 0$ and defining the map $\tilde{\psi} : T \to GL(E)$ by $\tilde{\psi}(\kappa) = \tilde{\psi}_E(\kappa, 0)$, we obtain $\tilde{F}(\kappa) \equiv \tilde{\psi}(\kappa) F(\kappa)$.

Combining the arguments of the preceding two paragraphs produces $\psi, \tilde{\psi}$ and the identity (3.6) in the general case. □

As one can see from the proof, when $\mathcal{X}$ and $\mathcal{Y}$ are endowed with Γ-actions, if Φ is further assumed to be Γ-equivariant, the Kuranishi map $F : T \to E$ is also Γ-equivariant because the implicit function theorem carries out without modification in the equivariant setting. If this procedure can be performed in a global fashion, meaning that $\Phi : \mathcal{X} \to \mathcal{Y}$ is actually a Fredholm section of a Banach vector bundle over a Banach manifold, the output in the Γ-equivariant setting is exactly a Γ-equivariant Kuranishi model.

4. Local Kuranishi model for stable maps

Let (M, J) be an almost complex manifold. Suppose (Σ, j) is a Riemann surface with at worst nodal singularities, i.e., locally modeled on $\{xy = 0\} \subset \mathbb{C}^2$. A smooth map $u : \Sigma \to M$ is defined to be a smooth map over the normalization of Σ (separating the branches at nodal points) whose takes the same value at the preimages of a given nodal point. It is said to be J-holomorphic if

$$du \circ j = J \circ du$$

holds over the normalization of Σ.

Definition 4.1. *A J-holomorphic map $u : (\Sigma, j) \to (M, J)$ is called* stable *if there exists only definitely many automorphisms $\phi : (\Sigma, j) \to (\Sigma, j)$ such that $u \circ \phi = u$.*

A Riemann surface (Σ, j) *with marked points is called stable if there are only finitely many automorphisms of* (Σ, j) *fixing the marked points.*

Let $\overline{\mathcal{M}}$ be the moduli space of J-holomorphic stable maps to M, and we have chosen to be deliberately vague with the additional data entering into the definition. We wish to find a *local Kuranishi model* near $u : (\Sigma, j) \to (M, J)$ for $[u] \in \overline{\mathcal{M}}$ in the following sense: we want to find a quadruple (Γ, V_1, V_2, f) where Γ is a finite group, V_1 and V_2 are finite-dimensional Γ-representations, $f : V_1 \to V_2$ is a Γ-equivariant continuous map, such that there exists a continuous map

$$\psi : f^{-1}(0)/\Gamma \to \mathcal{M}$$

which defines a homeomorphism onto an open subset of $\mathcal{M}$ containing $[u]$.

For the first step, we need a method to *stabilize the domain* of u such that after adding some marked points, (Σ, j) becomes stable as a Riemann surface.

Lemma 4.2. *Let* $u : (\Sigma, j) \to (M, J)$ *be a stable* J*-holomorphic map. Then there exists a codimension* 2 *submanifold with boundary* $D \subset M$ *such that* D *only intersects* u *away from the nodes with transverse intersectin points, and if* (Σ, j) *is equipped with marked points coming from the intersection points with* D*, the domain becomes stable.*

Proof. Note that if u is a stable map, an irreducible component is unstable only if u is not constant when restricted along it. Therefore, the differential du must be injective at a smooth point x over such an irreducible component. Then we can choose D to be (a disjoint union) of manifolds-with-boundary coming from a small disk in the normal direction of du_x. This ensures the stability of the domain if we choose sufficiently many submanifolds. □

Now suppose that (Σ, j) is a genus g Riemann surface with k marked points. Choosing a D in Lemma 4.2, let $r := \# u^{-1}(D)$. Ordering these r points requires a choice, which defines a $\mathrm{Sym}(r)$-torsor, the symmetric group of r-elements. In other words, the moduli space of genus g curves with $k + r$ marked points quotient out by the $\mathrm{Sym}(r)$-action

$$\overline{\mathcal{M}}_{g,k+r}/\mathrm{Sym}(r)$$

defines a local model for the possible domains of curves in $\mathcal{M}$ near $[u]$. The family of curves

$$\overline{\mathcal{C}}_{g,k+r}/\mathrm{Sym}(r) \to \overline{\mathcal{M}}_{g,k+r}/\mathrm{Sym}(r)$$

parametrized by this moduli space provides the stage for applying the Kuranishi reduction. Indeed, we can present $\overline{\mathcal{M}}_{g,k+r}/\mathrm{Sym}(r)$ locally as a quotient of a manifold

$$\mathcal{M}/\Gamma,$$

over which we have the Γ-equivariant family of curves $\mathcal{C} \to \mathcal{M}$.

For the final step, we note that the moduli space $\mathcal{M}$ near $[u]$ is the zero locus of the $\overline{\partial}$-section of the Banach bundle over

$$W^{k,p}(\mathcal{C}, M)$$

whose fiber at $((\Sigma', j'), v)$ is given by $W^{k-1,p}(\Sigma'; \Omega^{0,1}_{\Sigma',j'} \otimes v^*TM)$ modulo the Γ-action. Applying the Kuranishi reduction, we obtain our (Γ, V_1, V_2, f), where V_1 and V_2 are respectively the kernel and cokernel of the linearization $D_u\overline{\partial}$.

Remark 4.3. *Recall that the Kuranishi reduction is constructed by applying the implicit function theorem to the stabilization by* $\operatorname{coker}(D_u\overline{\partial})$*. This says that the extra piece of thickening we should take to ensure the manifold structure of the moduli space exactly comes from enlarging the domain by* $\operatorname{coker}(D_u\overline{\partial})$*. As a comparison, in the equivariant Pontryagin–Thom construction, we can obtain a genuine manifold from the equivariant homotopy group in general only by thickening up using an equivariant vector bundle.*

It is evident that if we could present $\mathcal{M}$ globally as $s^{-1}(0)/\Gamma$ for a Γ-equivariant Kuranishi chart (M, E, s), we can import algebraic topology to investigate the topology of moduli spaces of J-holomorphic maps.